How to Find Algebraic Relations?

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Algebraic Relations – Elementary Viewpoint

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Consequence: The set of all algebraic relations forms a radical ideal.

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The ideal of algebraic relations among $f_1(n), \ldots, f_m(n)$ is precisely the kernel of this map, $\ker\phi$.

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Summary:
$\{p \in \mathbb{K}[x_1, \dots, x_n] : p(f_1, \dots, f_m) \equiv 0\} = \ker \phi = I(P)$.

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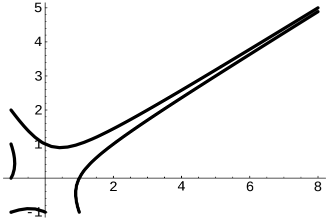

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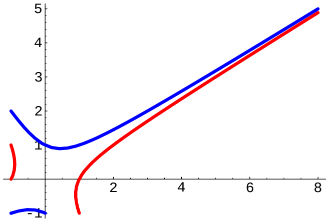

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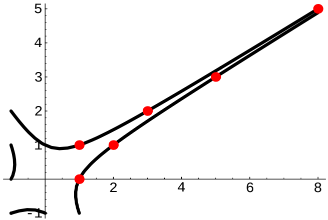

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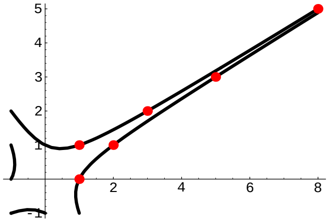

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Based on the geometric interpretation, it is straightforward to prove that $\mathfrak{a}$ is really the ideal claimed above.

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Problem: Given sequences $f_1(n), \ldots, f_m(n)$ and $p \in \mathbb{K}[x_1, \ldots, x_n]$, decide

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(e.g., gfun can do this for $f_i(n)$ P-finite.)

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Applications: Summation/Integration of special functions.
We have an algorithm that can find identities like

$$\sum_{k=0}^{n}((k-\sqrt{k}+1)H_k+1)\sqrt{k!}=(1+(n+1)H_n)\sqrt{n!}$$

which depend on exploiting the relation $\sqrt{n}^2 - n = 0$.

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Applications: Summation/Integration of special functions.
We want to have an algorithm that can find identities like

$$\sum_{k=0}^{n} \frac{((-1)^k-1)x+(-1)^k+1}{2U_k(x)+(-1)^k-1} = \frac{(1-2x)U_n(x)+(-1)^n+U_{n+1}(x)}{2U_n(x)+(-1)^n-1}$$

which also depend on nontrivial relations.

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Today we discuss finding.

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Of course, "given sequences" makes only sense when attention is restricted to particular *classes* of sequences that admit finitary representations, e.g., by defining recurrence equations.

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Consider the polynomial

$$p(x_1, x_2) = a_0 x_1^2 + a_1 x_1 x_2 + a_2 x_2^2 + a_3 x_1 + a_4 x_2 + a_5$$

with undetermined coefficients $a_0, a_1, \ldots, a_5$.

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For $p(x_1, x_2)$ to be an algebraic relation of $f_1(n), f_2(n)$, we must have

$$\forall n \geq 0 : p(f_1(n), f_2(n)) = 0$$

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\begin{aligned}
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&\vdots \\
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For speed-up, use the Buchberger-Möller algorithm.

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Note: We can determine all algebraic relations up to a prescribed degree, but we get no information about existence/non-existence of higher degree relations.

Example: Somos Sequences

A sequence C_n satisfying a nonlinear recurrence of the form

$$C_{n+r}C_n = \alpha_1 C_{n+r-1}C_{n+1} + \alpha_2 C_{n+r-2}C_{n+2} + \cdots$$
$$\cdots + \alpha_{\lfloor r/2 \rfloor} C_{n+r-\lfloor r/2 \rfloor} C_{n+\lfloor r/2 \rfloor}$$

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Example: Consider C_n defined via

$$C_{n+4}C_n = C_{n+3}C_{n+1} + C_{n+2}^2, \qquad C_0 = C_1 = C_2 = C_3 = 1.$$

Does this sequence satisfy a Somos-like recurrence of orders $5, 6, 7, 8$?

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Reduction modulo $\mathfrak{a}$ gives

$$x_5x_0 - a_1x_4x_1 - a_2x_3x_2 \longrightarrow_{\mathfrak{a}} (1 - \tfrac{1}{5}a_2)x_0x_5 - (a_1 + \tfrac{1}{5}a_2)x_1x_4$$

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Reduction modulo $\mathfrak{a}$ gives

$$x_5x_0 - a_1x_4x_1 - a_2x_3x_2 \longrightarrow_{\mathfrak{a}} (1 - \tfrac{1}{5}a_2)x_0x_5 - (a_1 + \tfrac{1}{5}a_2)x_1x_4$$

Comparing coefficients gives $a_1 = -1, a_2 = 5$.

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For sufficiently rich classes $\mathcal{C}$ there is no hope for such an algorithm.

If we insist in a complete algorithm, we have to focus on smaller classes.

C-Finite Sequences

(joint work with B. Zimmermann)

C-finite Sequences

Recall: $f(n)$ is C-finite if

$$f(n+r) = a_0 f(n) + a_1 f(n+1) + \cdots + a_{r-1} f(n+r-1)$$

for some *constants* $a_0, \ldots, a_{r-1} \in \mathbb{K}$.

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Recall: $f(n)$ is C-finite if and only if

$$f(n) = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n \qquad (n \geq 0)$$

where ϕ_i are the roots of the characteristic polynomial

$$x^r - a_0 - a_1 x - a_2 x^2 - \cdots - a_{r-1} x^{r-1}$$

and $p_i(n)$ is a polynomial whose degree is bounded by the multiplicity of the root ϕ_i.

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Relation to Summation-Theory

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If $\mathbb{F}$ is a difference field and t_1, t_2, t_3 are formal *sums* over $\mathbb{F}$, then a new sum $t_4 = \Sigma r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

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The theorem of the previous slide may be viewed as a corollary to the second case, with $\mathbb{F} = \mathbb{K}$ (constants) and the t_i being exponentials.

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The theorem of the previous slide may be viewed as a corollary to the second case, with $\mathbb{F} = \mathbb{K}$ (constants) and the t_i being exponentials. But it can be proven also by an elementary argument.

Arbitrary C-Finite Sequences

Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \trianglelefteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

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1. Write the sequences in the form

$$f_i(n) = p_{i,0}(n)\phi_1^n + \cdots + p_{i,l}(n)\phi_l^n$$

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$$\langle b_1, \ldots, b_r \rangle \trianglelefteq \mathbb{K}[y_0, y_1, \ldots, y_l]$$

of all relations among $n, \phi_1^n, \ldots, \phi_l^n$.

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For $\phi_j \in \mathbb{Q}$ this is easy.

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For $\phi_j \in \bar{\mathbb{Q}}$ this can be done with LLL and diophantine approximation (Ge's algorithm).

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3. Form the ideal

$$\begin{aligned}\mathfrak{a} := \big\langle & x_1 - (p_{1,0}(y_0)y_1 + \cdots + p_{1,l}(y_0)y_l),\\ & x_2 - (p_{2,0}(y_0)y_1 + \cdots + p_{2,l}(y_0)y_l),\\ & \ldots\\ & x_m - (p_{m,0}(y_0)y_1 + \cdots + p_{m,l}(y_0)y_l),\\ & b_1, \ldots, b_r\big\rangle \trianglelefteq \mathbb{K}[x_1, \ldots, x_m, y_0, \ldots, y_l]\end{aligned}$$

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These are precisely the desired relations.

Example: Fibonacci Numbers

Let F_n denote the nth Fibonacci number ($n \in \mathbb{Z}$).

> *Exercise 6.81:* (Graham/Knuth/Patashnik) *Let $P(x, y)$ be a polynomial in x and y with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.*

In other words: Find the ideal $\mathfrak{a} \trianglelefteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n\geq 0}$ and $(F_{n+1})_{n\geq 0}$.

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Note: We can determine *all* algebraic relations with this algorithm.

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$$F_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

$$F_{n+1} = \frac{5+\sqrt{5}}{10}\Big(\frac{1+\sqrt{5}}{2}\Big)^n + \frac{5-\sqrt{5}}{10}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

Example: Fibonacci Numbers

en detail:

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So $\phi_1 = \frac{1}{2}(1+\sqrt{5})$ and $\phi_2 = \frac{1}{2}(1-\sqrt{5})$.

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$$\langle y_1^2 y_2^2 - 1 \rangle \trianglelefteq \bar{\mathbb{Q}}[y_1, \ldots, y_l]$$

are the relations among n, ϕ_1^n, ϕ_2^n.

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$$F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1.$$

3. Next we set

$$\begin{aligned}\mathfrak{a} := \big\langle x_1 - \big(\tfrac{1}{\sqrt{5}} y_1 - \tfrac{1}{\sqrt{5}} y_2\big),\\ x_2 - \big(\tfrac{5+\sqrt{5}}{10} y_1 + \tfrac{5-\sqrt{5}}{10} y_2\big),\\ y_1^2 y_2^2 - 1\big\rangle \trianglelefteq \bar{\mathbb{K}}[x_1, x_2, y_0, y_1, y_2]\end{aligned}$$

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Note: Intermediate algebraic field extensions always cancel out in the final result.

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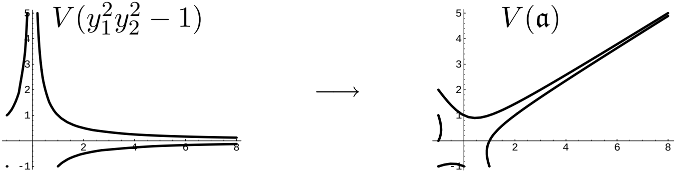

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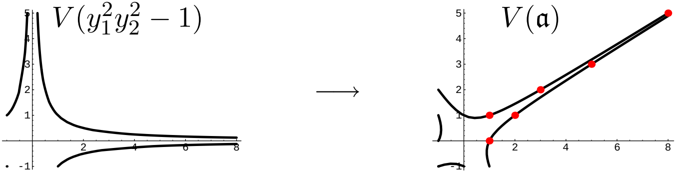

Multivariate Sequences

A sequence $f\colon \mathbb{Z}^d \to \mathbb{C}$ is *multi-C-finite* if it satisfies a C-finite recurrence equation in every direction.

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For example, if $f(n,m)$ is such that $f(0,0) = f(0,1) = 1$ and

$$\begin{aligned} f(n+2,m) &= 4f(n+1,m) - 4f(n,m) \\ f(n,m+1) &= 5f(n,m), \end{aligned}$$

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Algebraic relations among multi-C-finite sequences can be found in very much the same way as for univariate sequences.

Some Funny Applications

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Proving Identities

For deciding $p(F_n, F_{n+1}) = 0$ for a given polynomial p, compute a normal form of p wrt. a Gröbner basis of the ideal of algebraic relations.

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(This makes only sense if you have many p for the same sequences.)

Some Funny Applications

"Express something in terms of something else"

Given $f(n)$ and $g_1(n), \ldots, g_m(n)$, is there a formula

$$f(n) = A\big(g_1(n), \ldots, g_m(n)\big) \quad (n \geq 0)$$

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Inspect polynomial with least degree with respect to that variable appearing in the Gröbner basis.

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Let $f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$. (This is C-finite, order 2)

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Consequently,

$$f(n) = F_n(2F_n^2 - 3F_nF_{n+1} + 3F_{n+1}^2).$$

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Consequently, the sum $f(n)$ *has no closed form* in terms of Fibonacci numbers.

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The second gives another identity for free: $F_{3n+1} F_{n+1} \equiv 1 \bmod F_n$.

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$$\mathfrak{a} = \langle x_1^4 - 10x_1^3 x_2 + 35x_1^2 x_2 - 50x_1 x_2^3 + 25x_2^4 - 1 \rangle.$$

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Next, use Gröbner bases to determine u, v, w with

$$1 = ux_1 + vx_2 + w(x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1).$$

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It follows that $1 = p(n)L_n + q(n)F_{n+1}$ for some $p(n), q(n) \in \mathbb{Z}$.

Therefore $\gcd(L_n, F_{n+1}) \mid 1$. This suffices.

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- There is an algorithm which computes the algebraic relations among some given C-finite sequences.
- All these relations are consequences of multiplicative relations among the roots of the characteristic polynomial.
- A lattice basis for these relations can be computed with a number-theoretic algorithm. The rest can be done with Gröbner bases.

What's next?

P-finite sequences?

Generating Functions

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- The algorithm presented earlier uses multiplicative relations among the singularities of the generating functions in order to make all the singularities cancel.
- Does this only work for rational generating functions (i.e. C-finite sequences)?
- Consider some examples. . .

Example 1

Let $f(n)$ be defined by

$$\begin{aligned}
&4(2n+3)(4n^2-1)\,f(n) + 4(2n+3)(n+1)\,f(n+1) \\
&\quad - (n+1)(n+2)(2n-3)\,f(n+2) = 0, \\
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\end{aligned}$$

There exist nontrivial algebraic relations among $f(n)$ and $f(n+1)$, e.g.,

$$\begin{aligned}
&((4n+2)f(n) - (n-4)f(n+1)) \\
&\quad \times ((4n^2-10n-6)f(n) - n(n+1)f(n+1)) = 0.
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This is in accordance with the generating function

$$\sum_{n=0}^{\infty} f(n)z^n = \tfrac{1}{12}\Big(\frac{i}{(4z-1)^{3/2}}+\frac{5i}{\sqrt{4z-1}}-\frac{4}{\sqrt{4z+1}}\Big)$$

having the singularities $+\frac{1}{4}$ and $-\frac{1}{4}$, which have a multiplicative relation.

Example 2

Let $f(n)$ be defined by

$$\begin{aligned} &2n(n+3)f(n) - (3n+8)(n+1)f(n+1) \\ &\qquad + (3n+8)(n+2)f(n+2) - (n+3)(n+2)f(n+3) = 0, \\ &f(0) = 1, f(1) = 3, f(2) = \tfrac{9}{2}. \end{aligned}$$

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There are nontrivial relations among $f(n), f(n+1), f(n+2)$ (too long to fit on this slide).

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This is in accordance with the generating function

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having the singularities $\frac{1}{2}$ and $(1+i\sqrt{3})/2$, $(1-i\sqrt{3})/2$, the latter two bearing a multiplicative relation.

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There do not seem to exist algebraic relations among $f(n)$, $f(n+1)$.

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Let $f(n)$ be defined as

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where $H_n(x)$ is the nth Hermite polynomial.

This is consistent with the generating function

$$\sum_{n=0}^{\infty} f(n) z^n = \exp(2xz - z^2)$$

having no singularities.

Can we construct
a complete algorithm
for finding algebraic relations
among P-finite sequences
using singularity analysis?