Enumeration and uniform sampling of planar structures

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Planar structures

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- . . .

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- Trees : K_3 minor-free graphs
- Planar graphs : K_5, $K_{3,3}$ minor-free graphs (Kuratowski's theorem)
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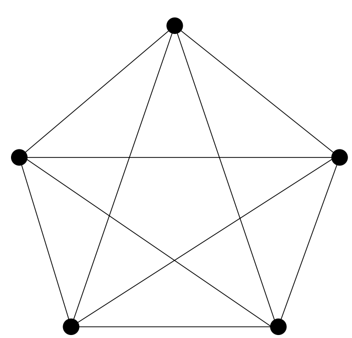

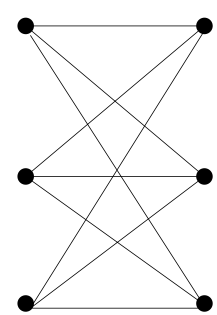

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  - average case analysis
  - empirical properties

Scheme

Decomposition

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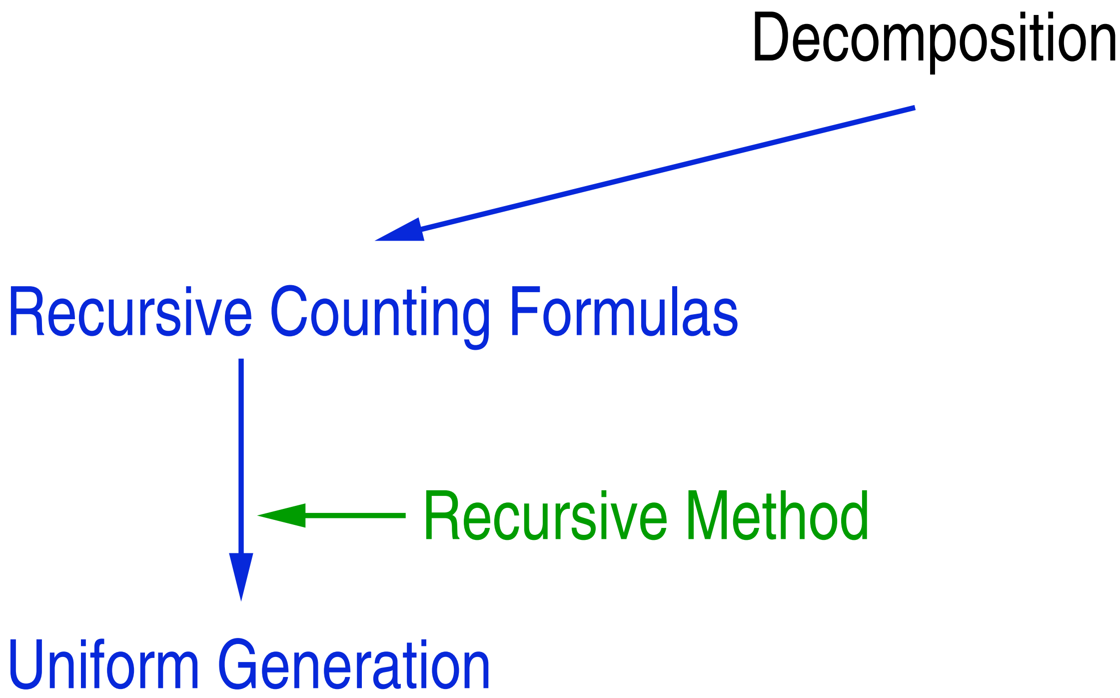
Recursive Counting Formulas

Scheme


Decomposition
Recursive Counting Formulas
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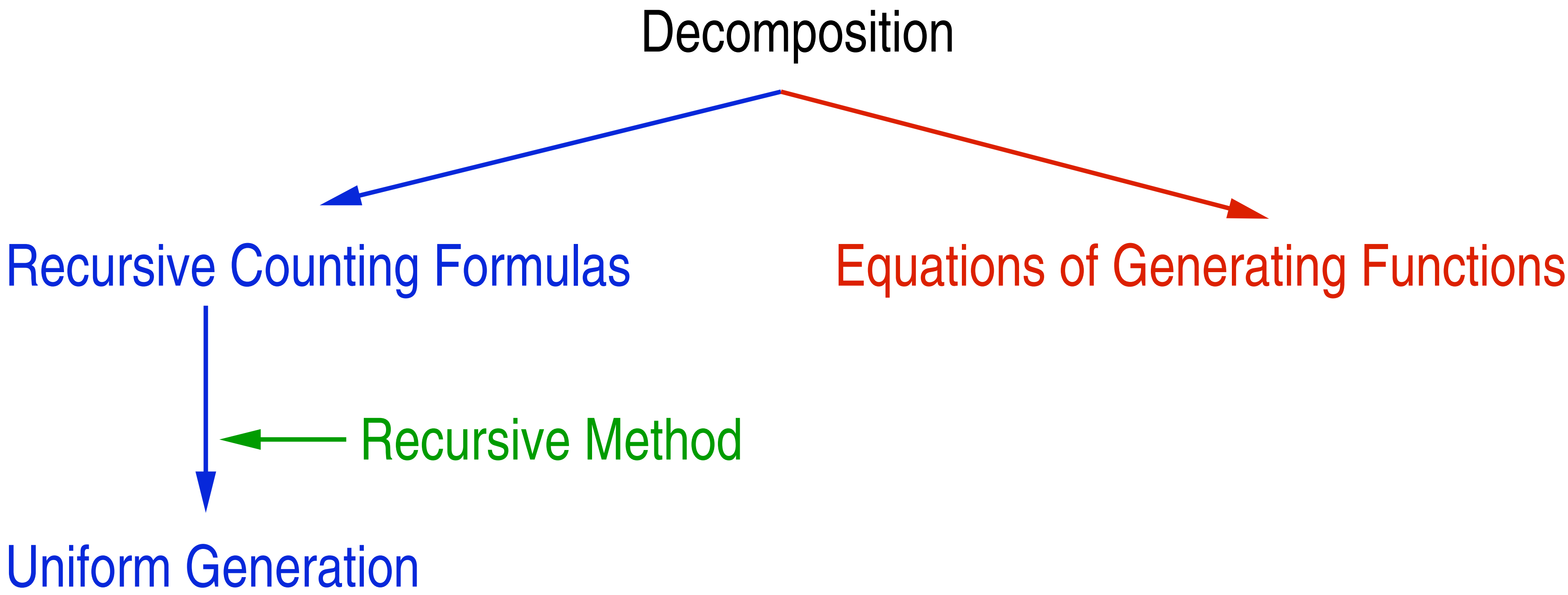
Scheme



Decomposition
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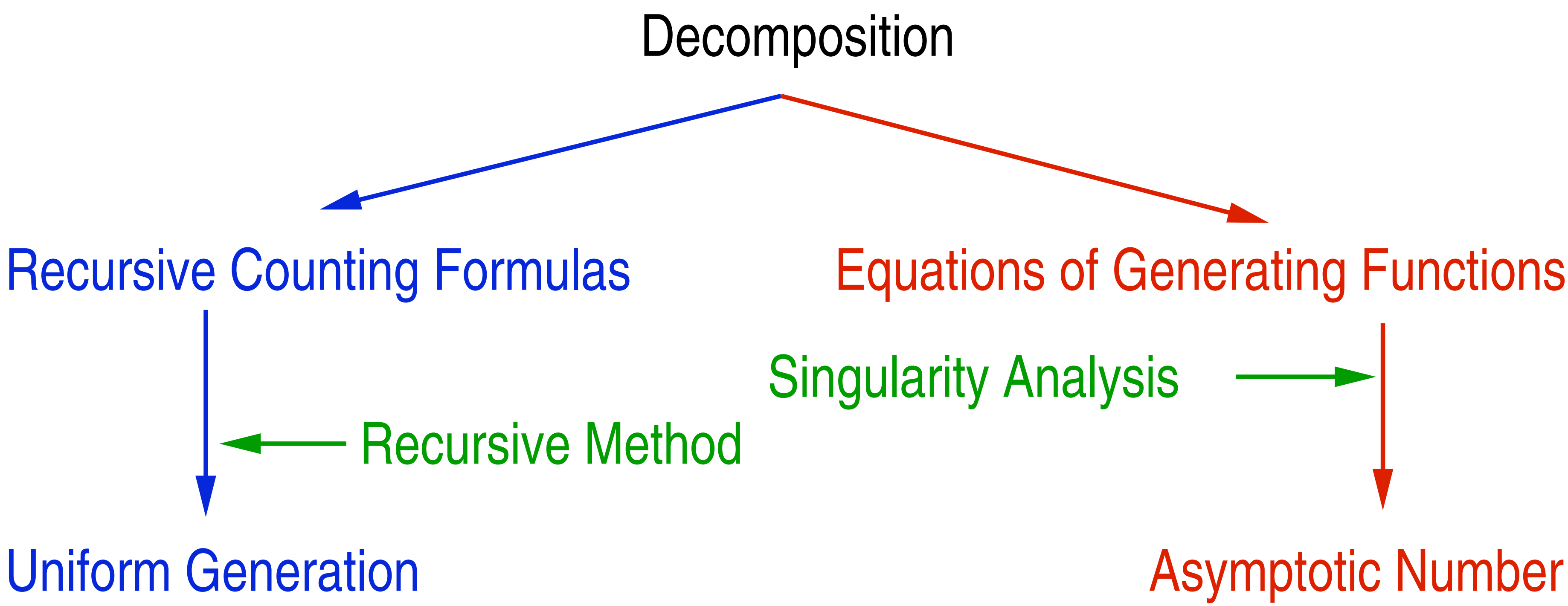
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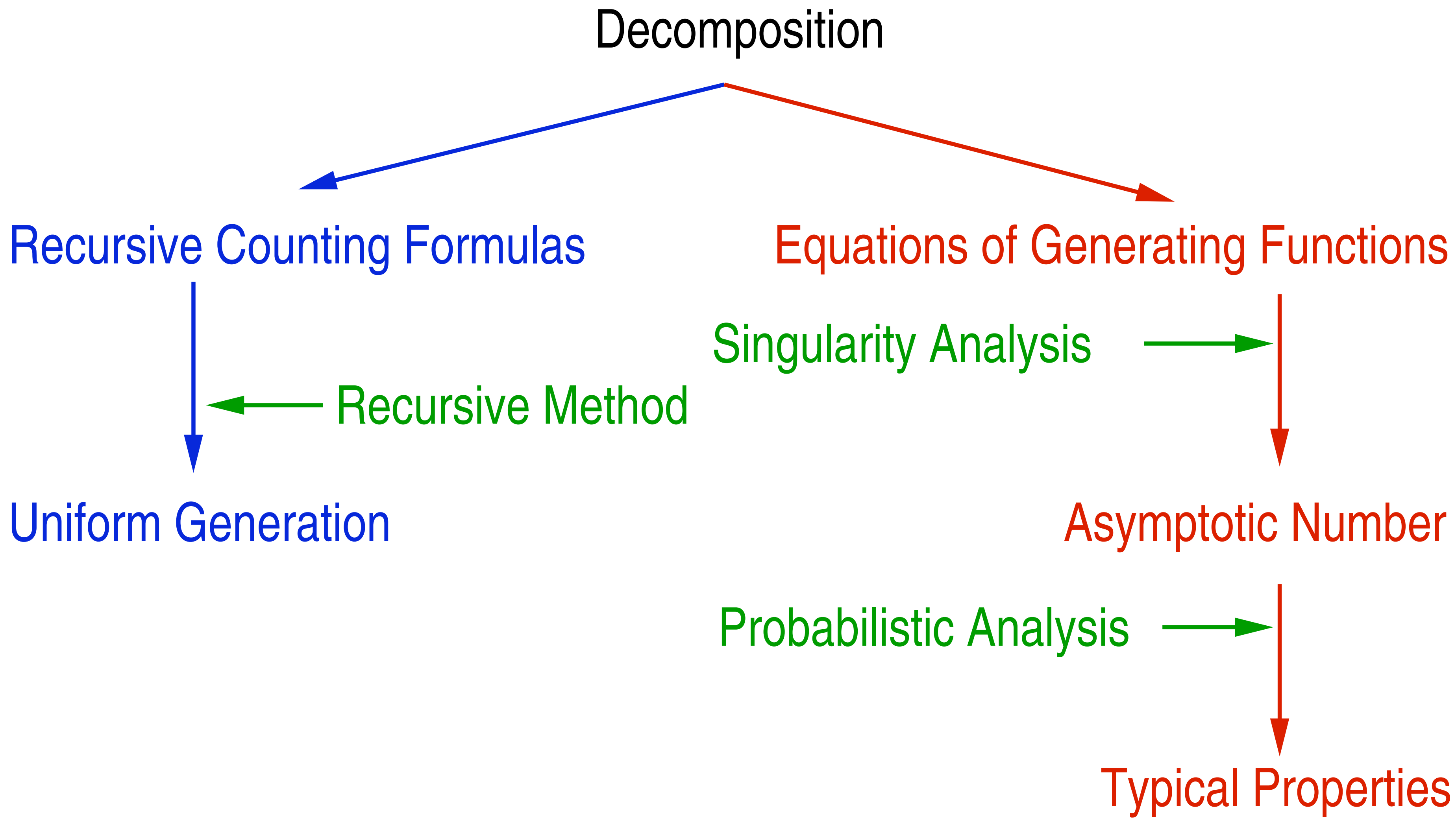
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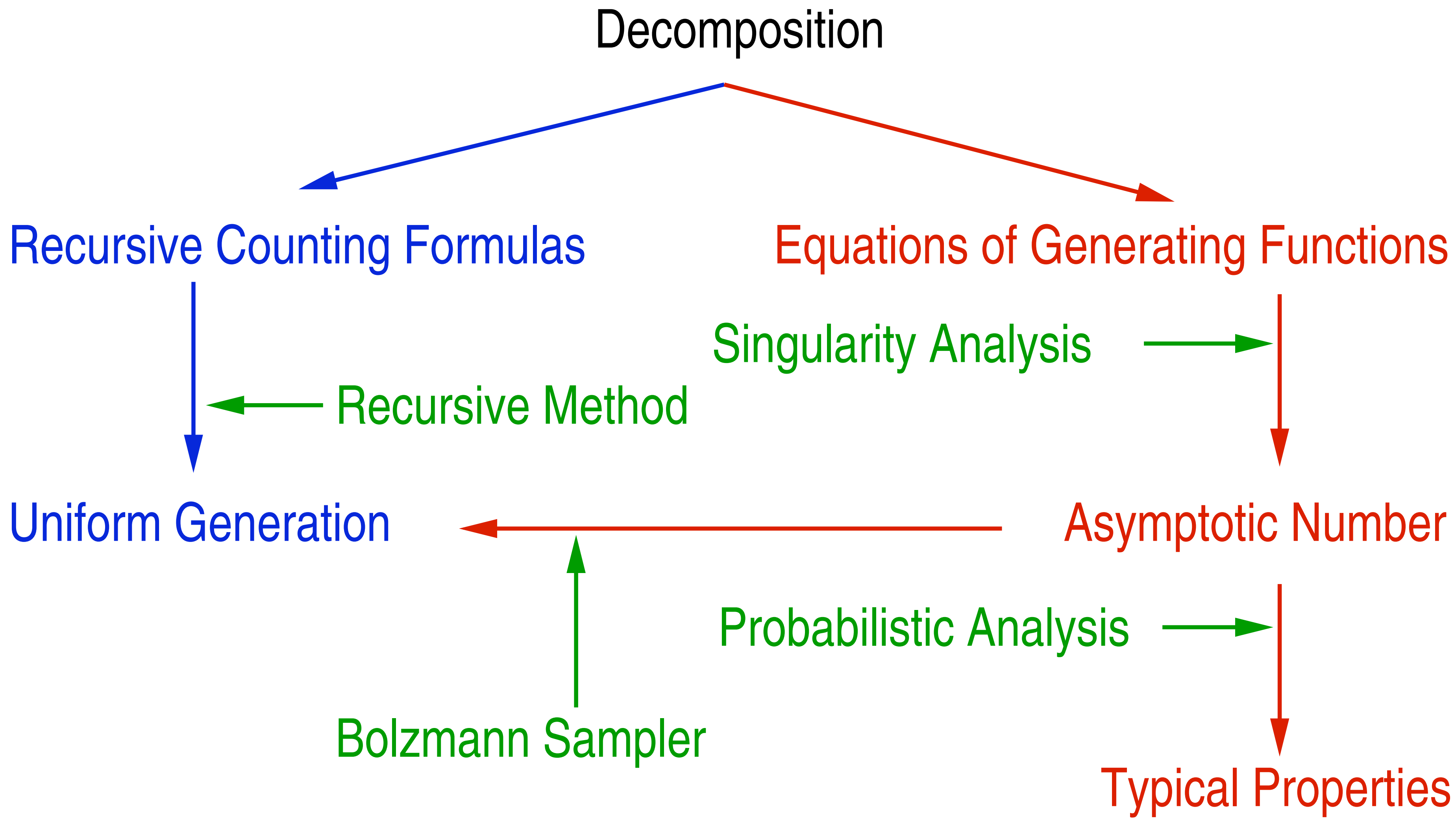
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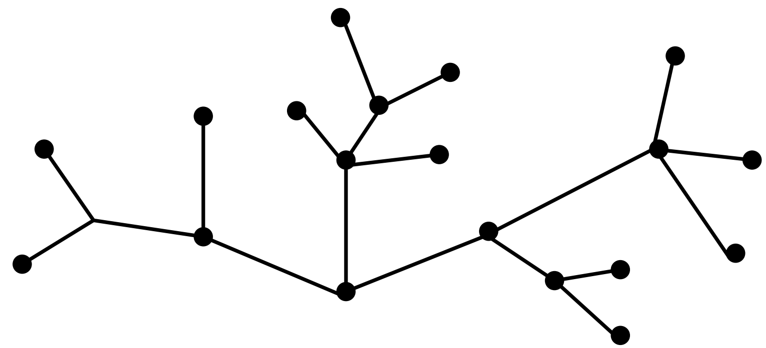
Labeled trees

How many labeled trees are there on vertex set $[n] := \{1, \cdots, n\}$?



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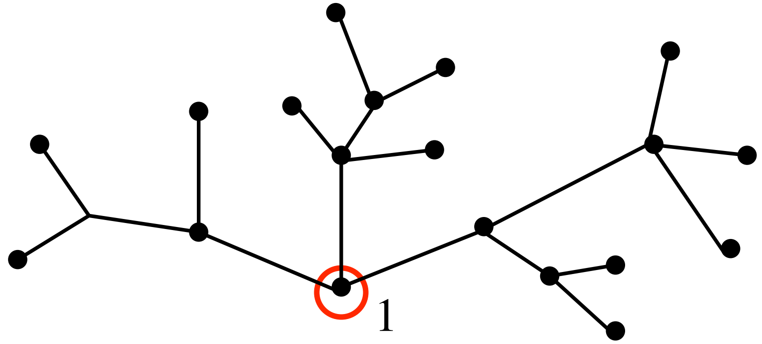

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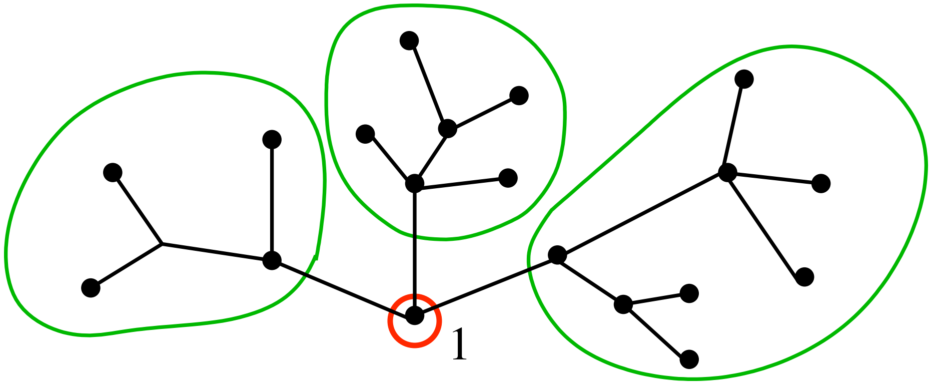

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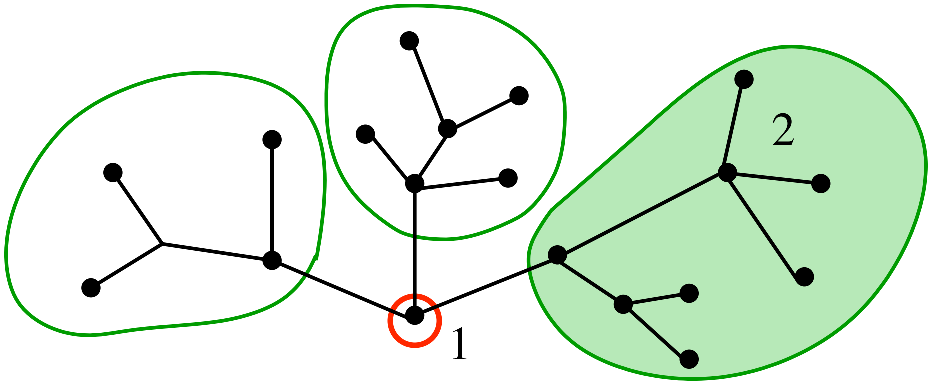

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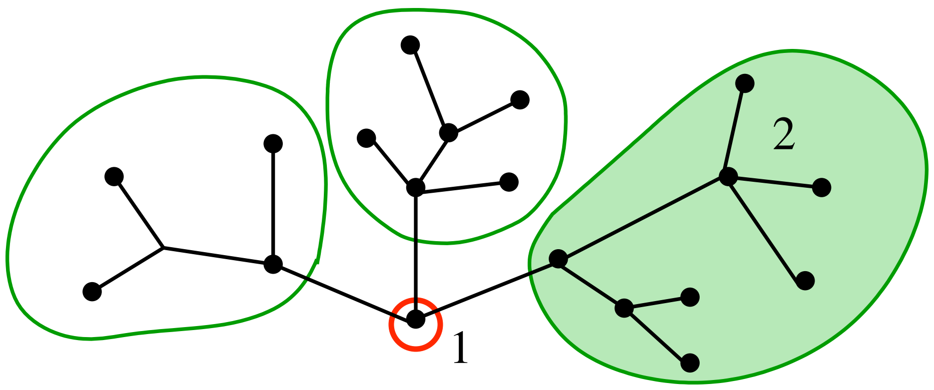

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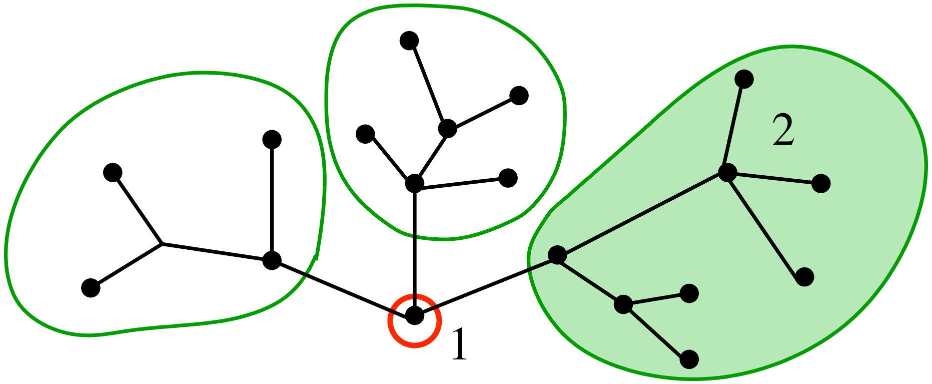

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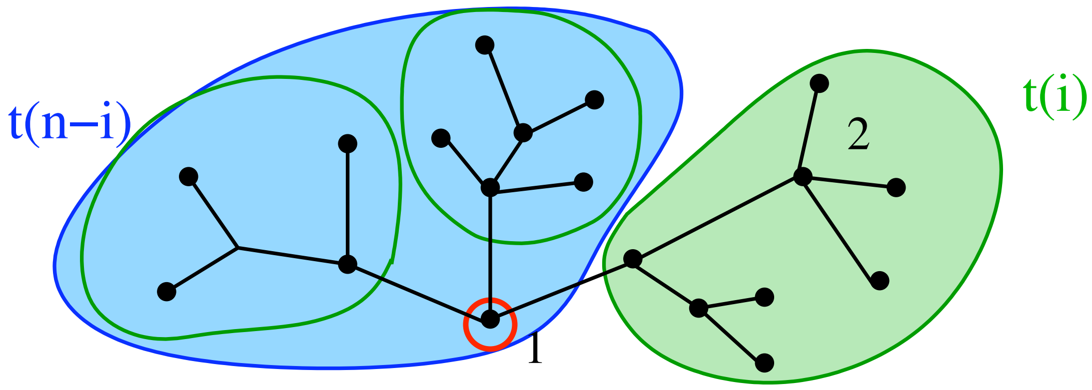

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$$\frac{t(n)}{n} = \sum_{i} \binom{n-2}{i-1} t(i) \frac{t(n-i)}{n-i}$$

Recursive method

[Nijenhuis, Wilf 79; Flajolet, Zimmerman, Van Cutsem 94]

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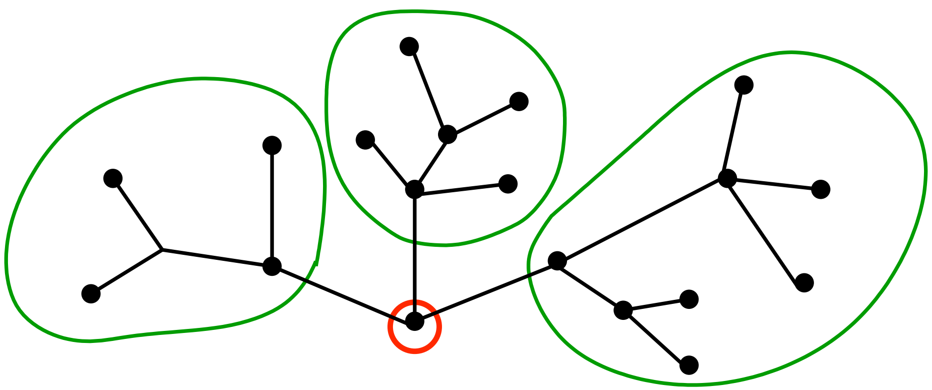
Uniform sampling algorithm for trees:

Generate(n): returns a random tree on $[n]$.
choose a root vertex r with probability $1/n$
return Generate(n, r)

Generate(n, r): returns a random tree on $[n]$ with the root vertex r
choose the order i of the split subtree with probability $n\binom{n-2}{i-1}t(i)t(n-i)/((n-i)t(n))$
let $s = \min([n] \setminus \{r\})$
choose a random subset $\{s\} \subseteq \{w_1, \ldots, w_i\} \subseteq [n] \setminus \{r\}$ (with relative order)
let $\{v_1, \ldots, v_{n-i}\} = [n] \setminus \{w_1, \ldots, w_i\}$ (with relative order)
$T_1 =$ **Generate**(i); relabel vertex j in T_1 with w_j (denote by r' the root vertex of T_1)
$T_2 =$ **Generate**$(n-i, r)$; relabel vertex $j \neq r$ in T_2 with v_j
return $T_1 \cup T_2 \cup \{(r, w_{r'})\}$ **with marked** r

Generating function



Let $T(z)$ be the exponential generating function for rooted trees defined by

$$T(z) = \sum_n t(n) \frac{z^n}{n!}.$$

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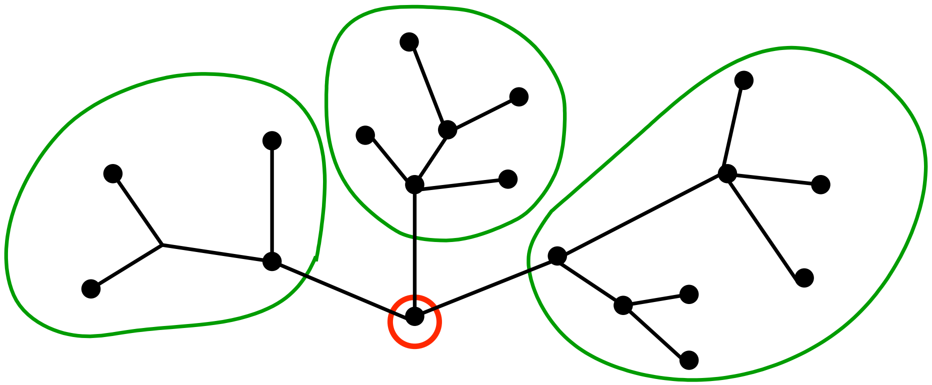

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$$T(z) = z\left(\qquad \frac{T(z)^3}{3!} \qquad \right)$$

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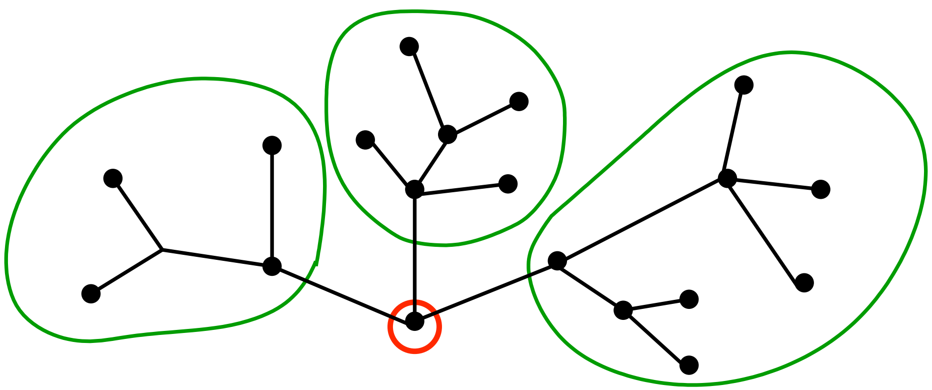

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Then

$$T(z) = {\color{red} z} \left(1 + T(z) + \frac{T(z)^2}{2!} + {\color{green} \frac{T(z)^3}{3!}} + \cdots \right)$$

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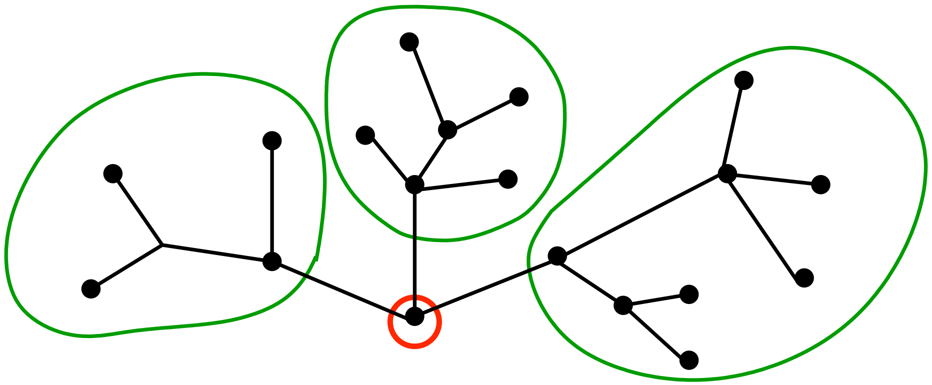

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$$T(z) = z\left(1 + T(z) + \frac{T(z)^2}{2!} + \frac{T(z)^3}{3!} + \cdots\right) = ze^{T(z)}.$$

LAGRANGE INVERSION THEOREM [FLAJOLET, SEDGEWICK 07+]

Let $\phi(u) = \sum_k \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then the equation

$$y = z\phi(y)$$

admits a unique solution in $\mathbb{C}[[z]]$ whose coefficients are given by

$$y(z) = \sum_n y_n z^n \quad \text{where} \quad y_n = \frac{1}{n}[u^{n-1}]\phi(u)^n\,.$$

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Thus the number of labeled trees on n vertices equals $\frac{t(n)}{n} = n^{n-2}$.

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How to determine

- the dominant singularity R and
- the subexponential factor $\theta(n)$?

Singularity analysis

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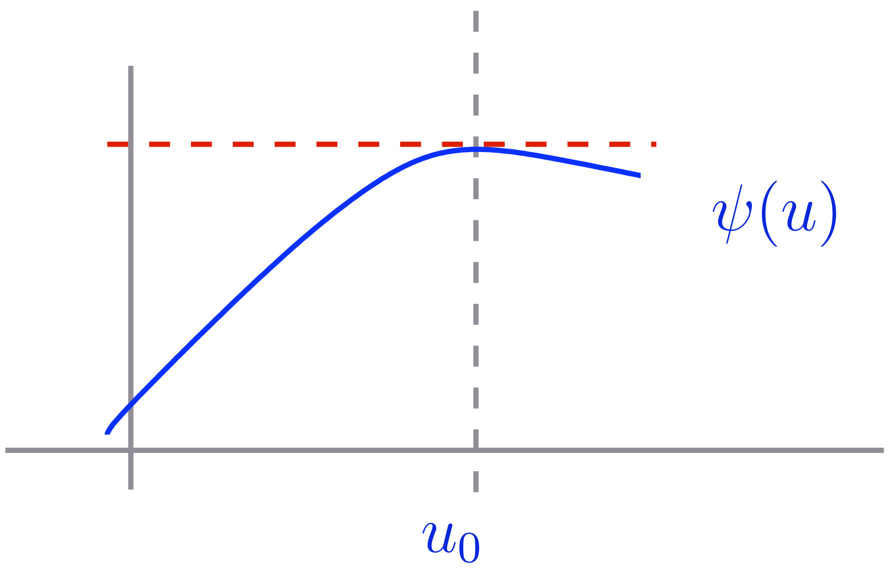

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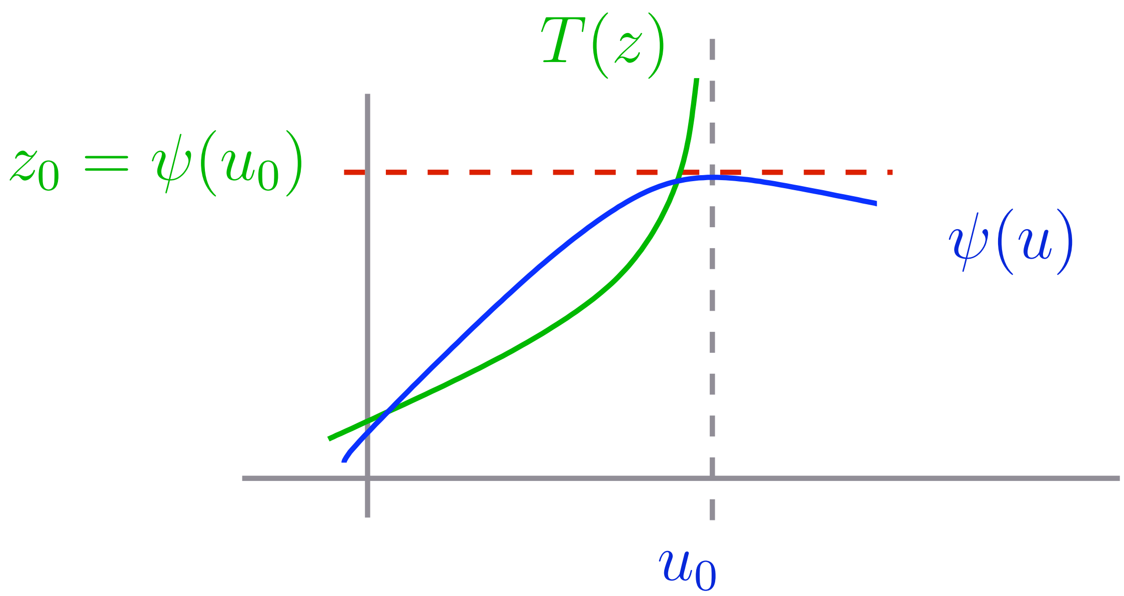

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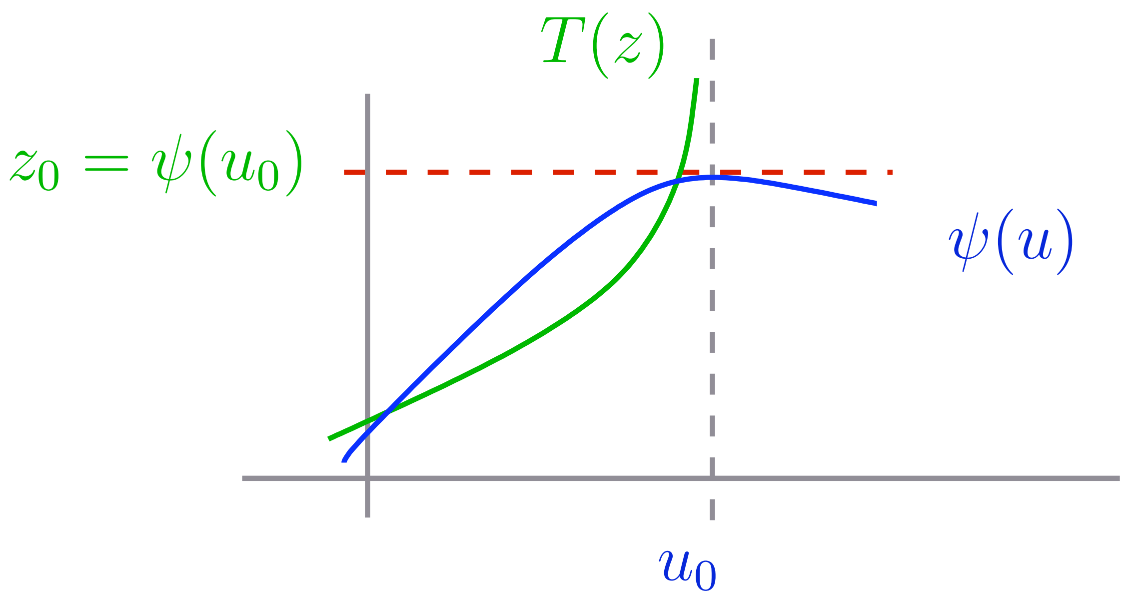

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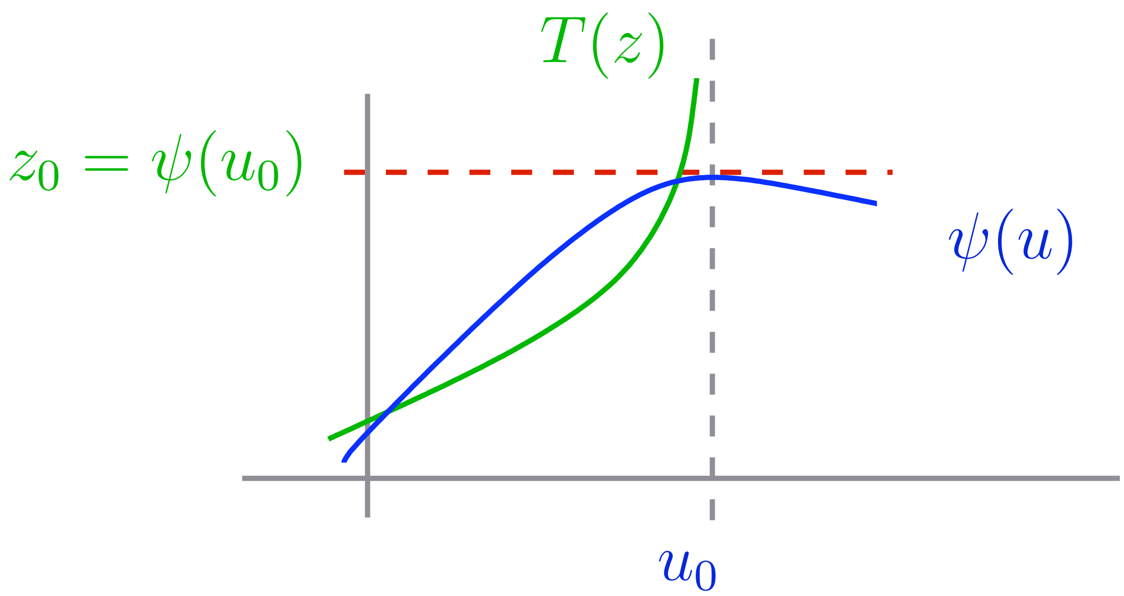

Indeed, $z_0 = e^{-1}$ and thus $\frac{t(n)}{n!} = \theta(n)e^n$, where $\limsup |\theta(n)|^{1/n} = 1$.

Local dependency

[Flajolet, Sedgewick 07+]

Taylor expansion of $z = \psi(u)$ at u_0 is of the form

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Using Δ-analycity of $T(z)$ and transfer theorem, we have

$$[z^n]T(z) \sim -\sqrt{-2\psi(u_0)/\psi''(u_0)}[z^n](1 - z/z_0)^{1/2}$$

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Rescaling rule/ generalized binomial theorem

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$$\begin{aligned} t(n) &\sim \frac{1}{\sqrt{2\pi}} n^{-3/2} e^n n! \\ &\sim \frac{1}{\sqrt{2\pi}} n^{-3/2} e^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \qquad \text{(Stirling's formula)} \end{aligned}$$

Basic scale

[Flajolet, Sedgewick 07+]

We have $z_0 = e^{-1}$, $u_0 = 1$, $\psi(u) = ue^{-u}$ and

$$[z^n]T(z) \sim -\sqrt{-2\psi(u_0)/\psi''(u_0)}[z^n](1 - z/z_0)^{1/2}.$$

Rescaling rule/ generalized binomial theorem

$$[z^n]\,(1 - z/z_0)^{1/2} = \binom{n - 3/2}{n} z_0^{-n} \sim \frac{n^{-3/2}}{-2\sqrt{\pi}} z_0^{-n}.$$

We have that the number of rooted trees on n vertices equals

$$\begin{aligned} t(n) &\sim \frac{1}{\sqrt{2\pi}} n^{-3/2} e^n n! \\ &\sim \frac{1}{\sqrt{2\pi}} n^{-3/2} e^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n} && \text{(Stirling's formula)} \\ &= n^{n-1} && \text{(Cayley's formula)} \end{aligned}$$

Block structure of a graph

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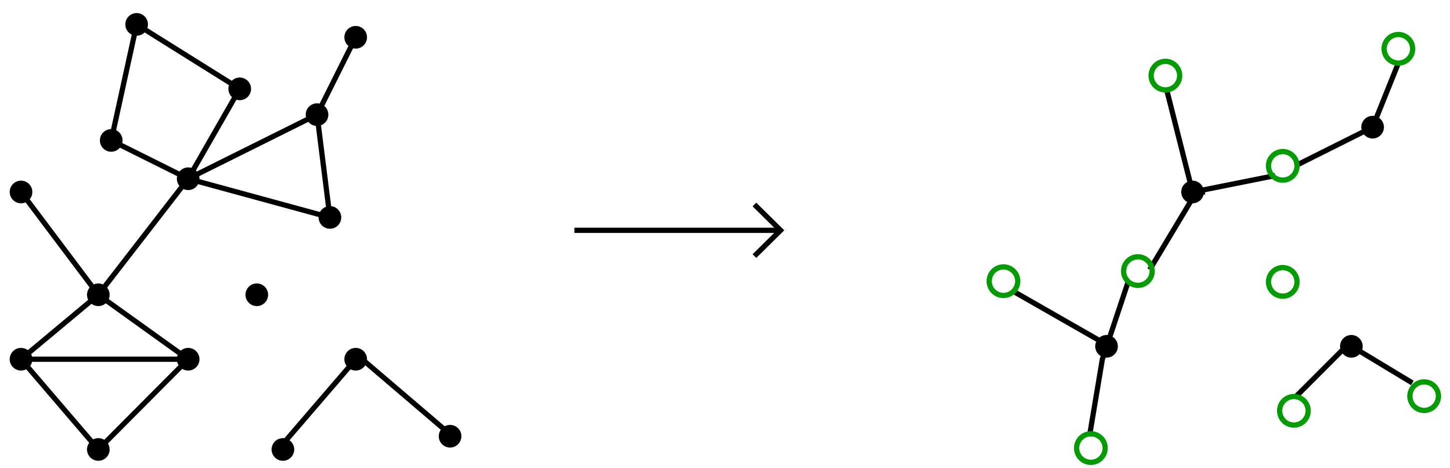
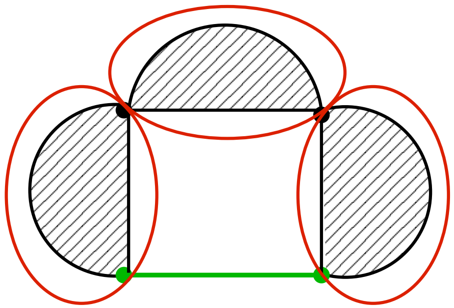
The block structure of a graph is a forest with two types of vertices: the blocks and the cutvertices of the graph.



Blocks of planar structures

2-connected outerplanar graphs:



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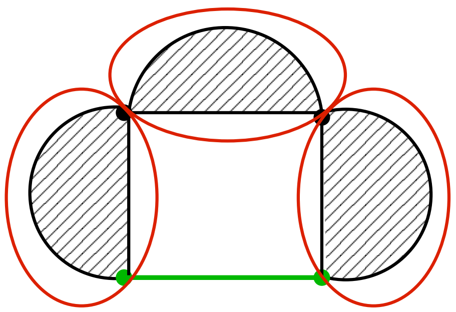

[Bodirsky, Giménez, K., Noy 07+]

outerplanar graphs on n vertices $\sim \alpha\, n^{-5/2}\, \rho^n\, n!\,, \quad \rho \doteq 7.32$

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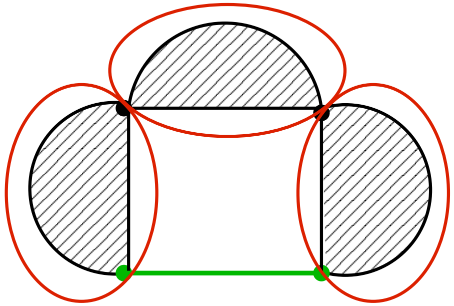

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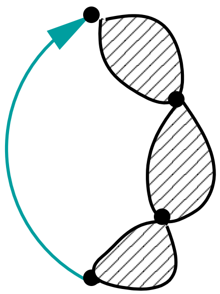

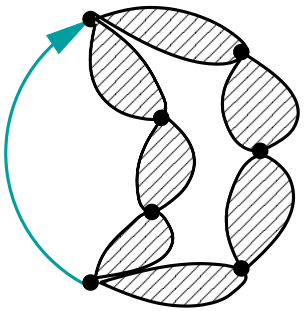

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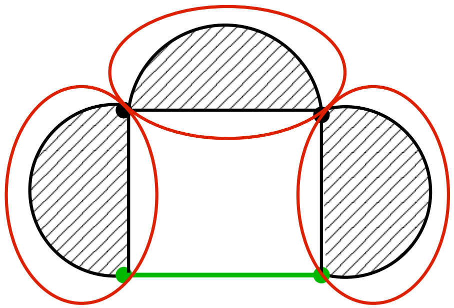

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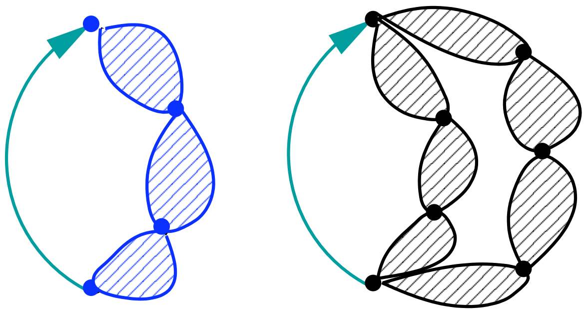

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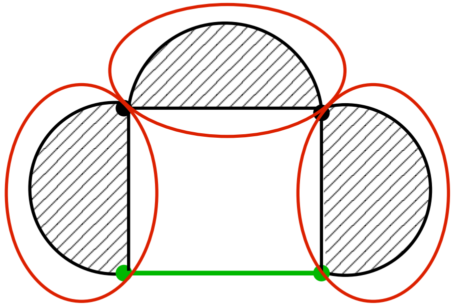

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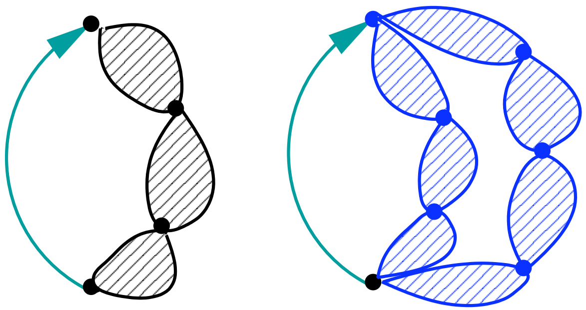

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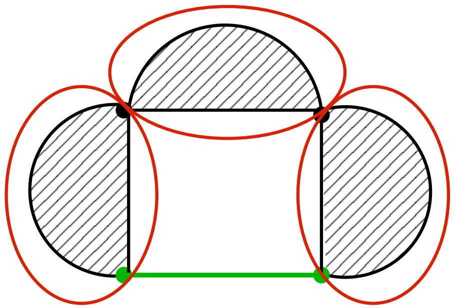

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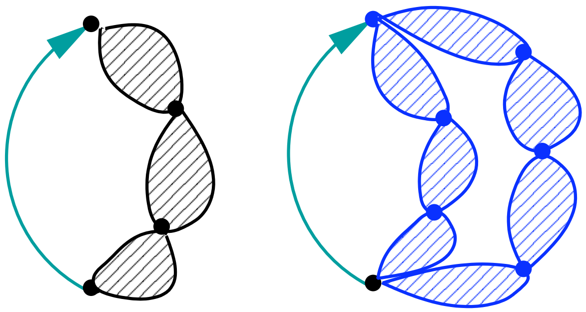

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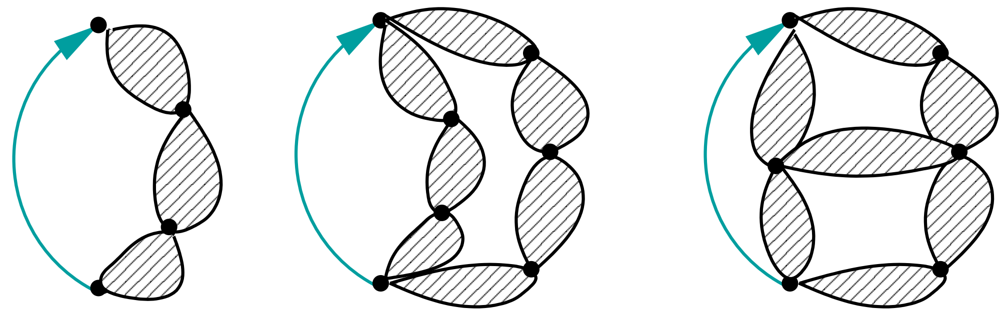
Labeled planar graphs

2-connected graphs

[TRAKHTENBROT 58; TUTTE 63; WALSH 82]



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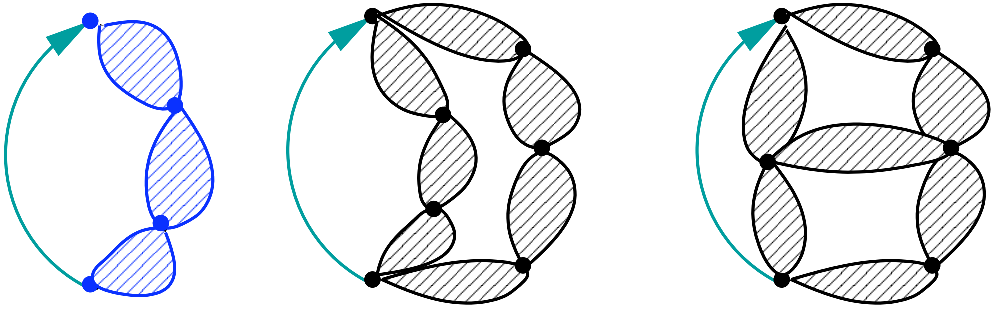

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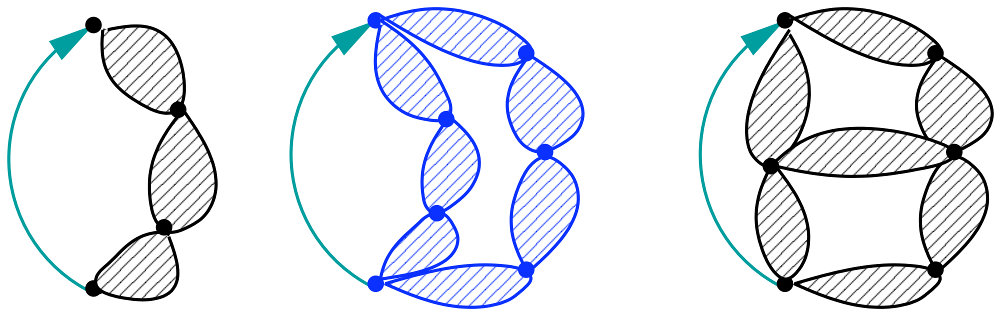

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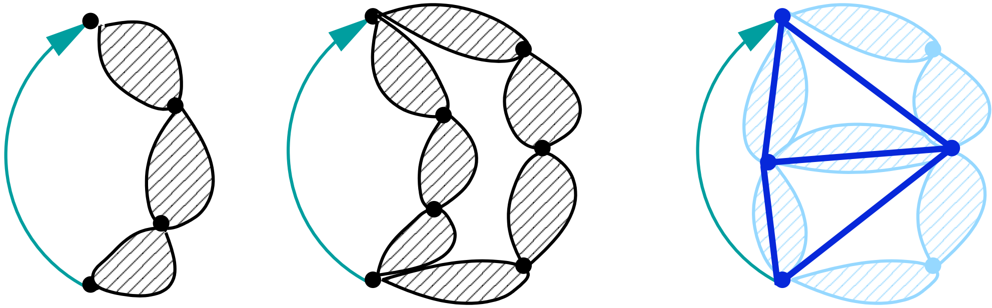

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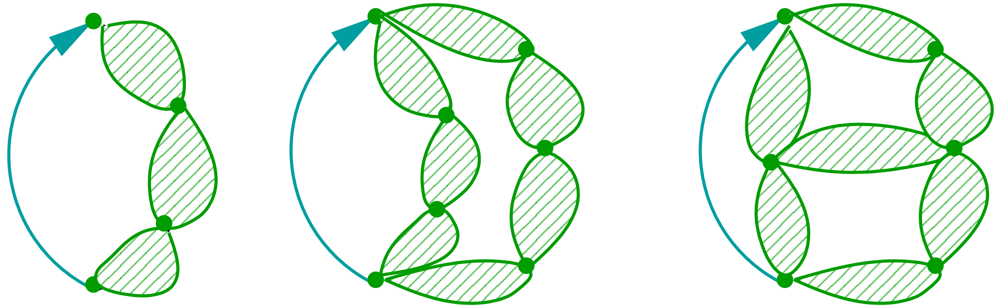

[Bender, Gao, Wormald 02]

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Labeled planar graphs

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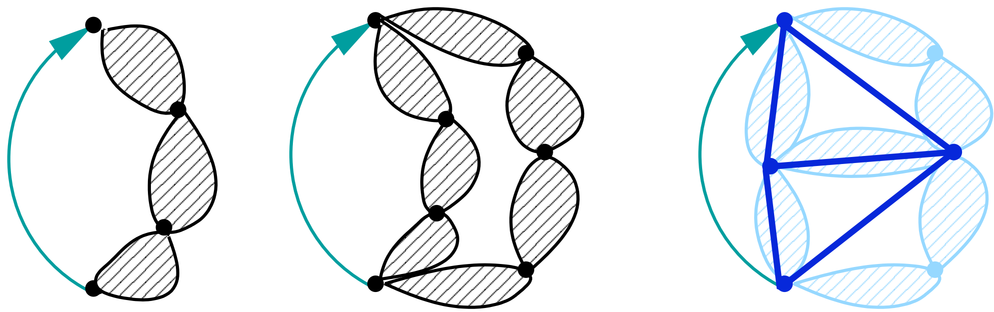

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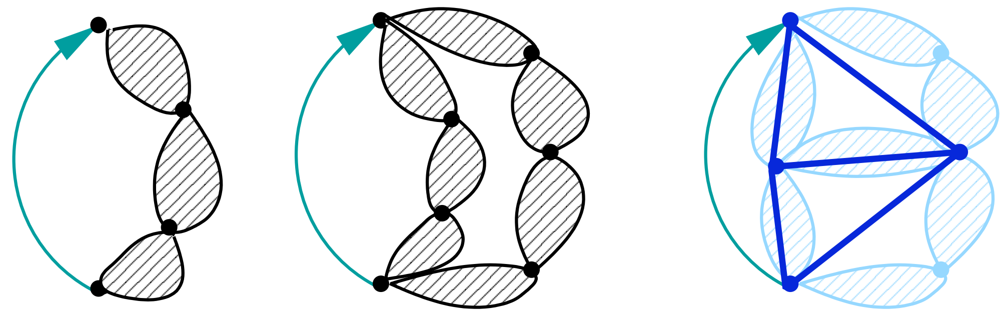

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Uniform sampling algorithm

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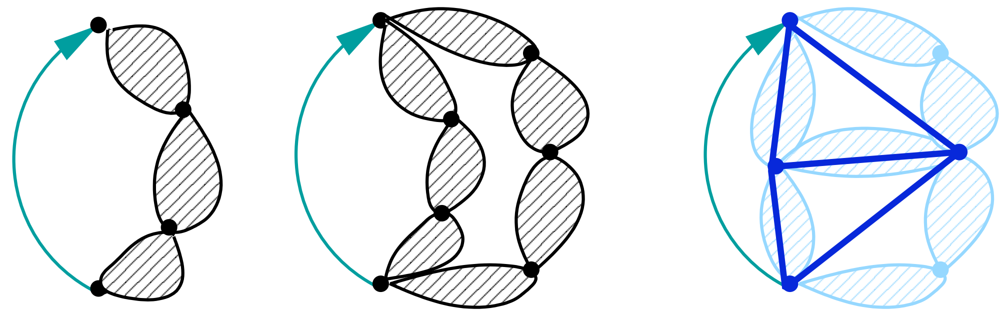

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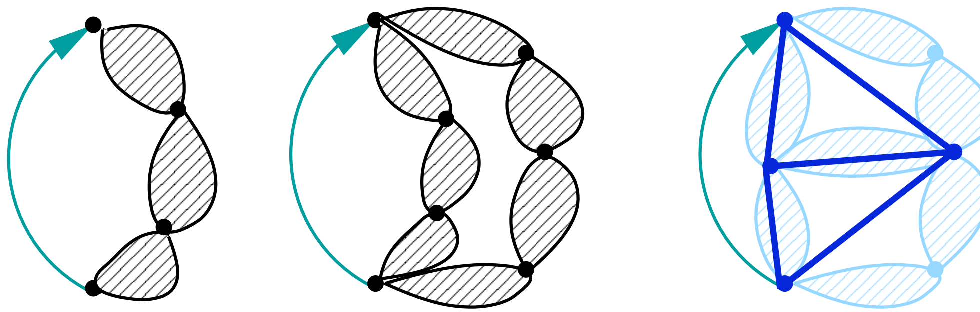

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Uniform sampling algorithm for planar graphs $O(n^7)$

Labeled planar graphs

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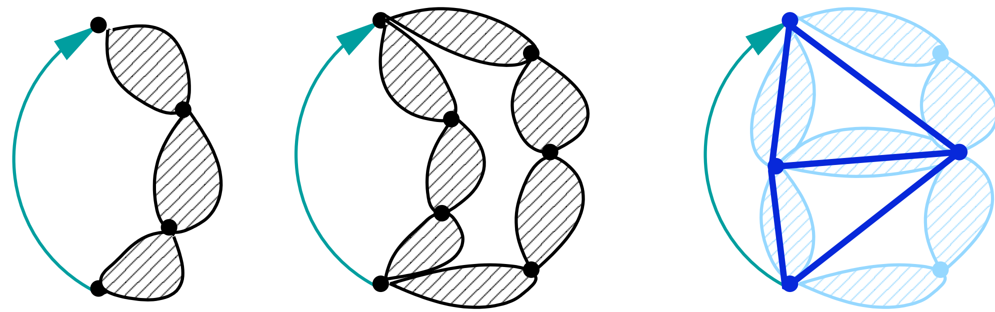

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The number of planar graphs is $\sim c\, n^{-7/2}\, 27.22^n n!$

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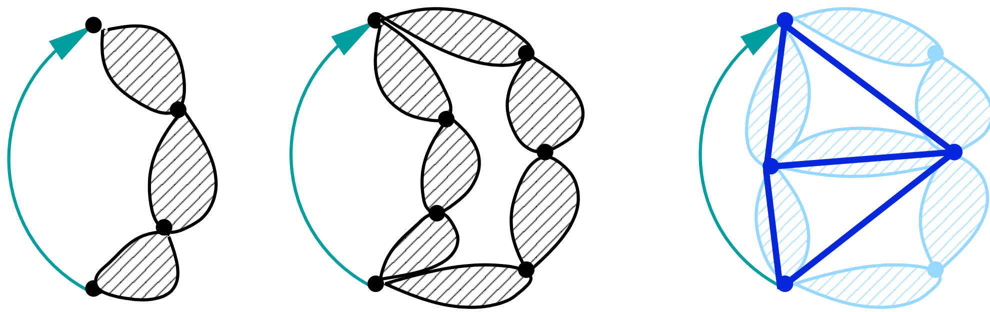

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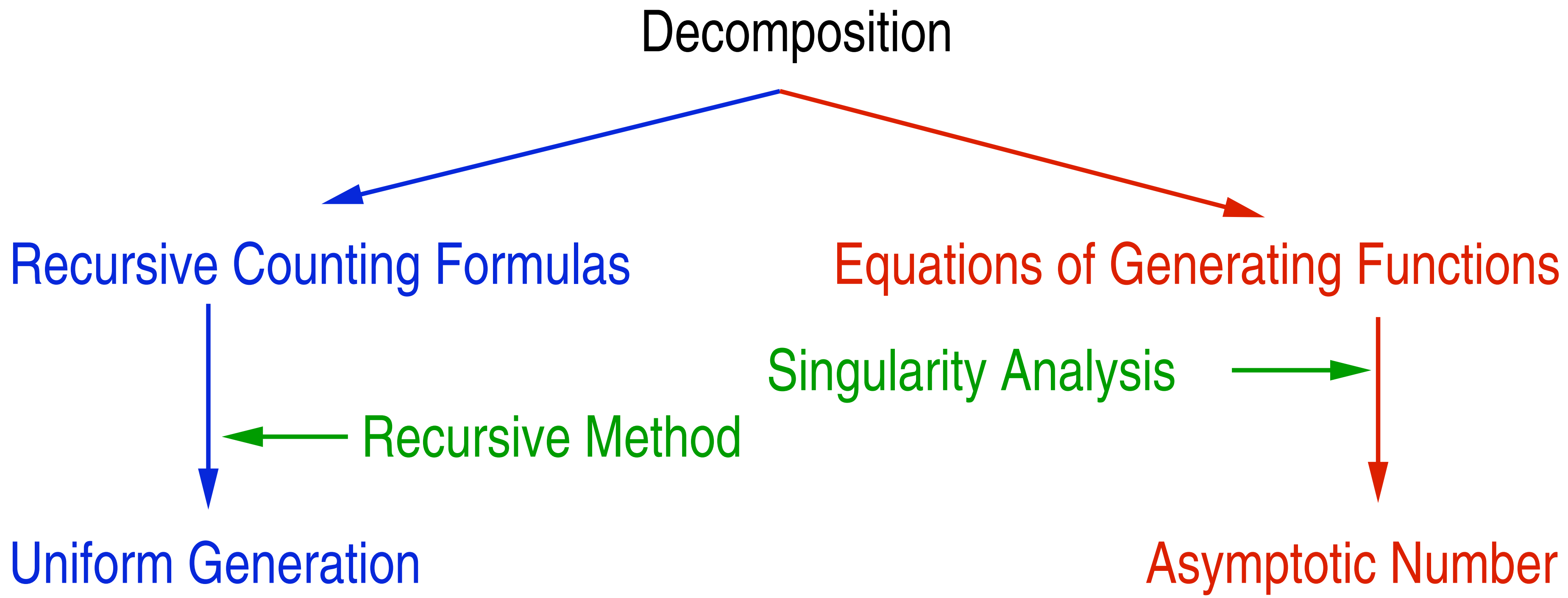
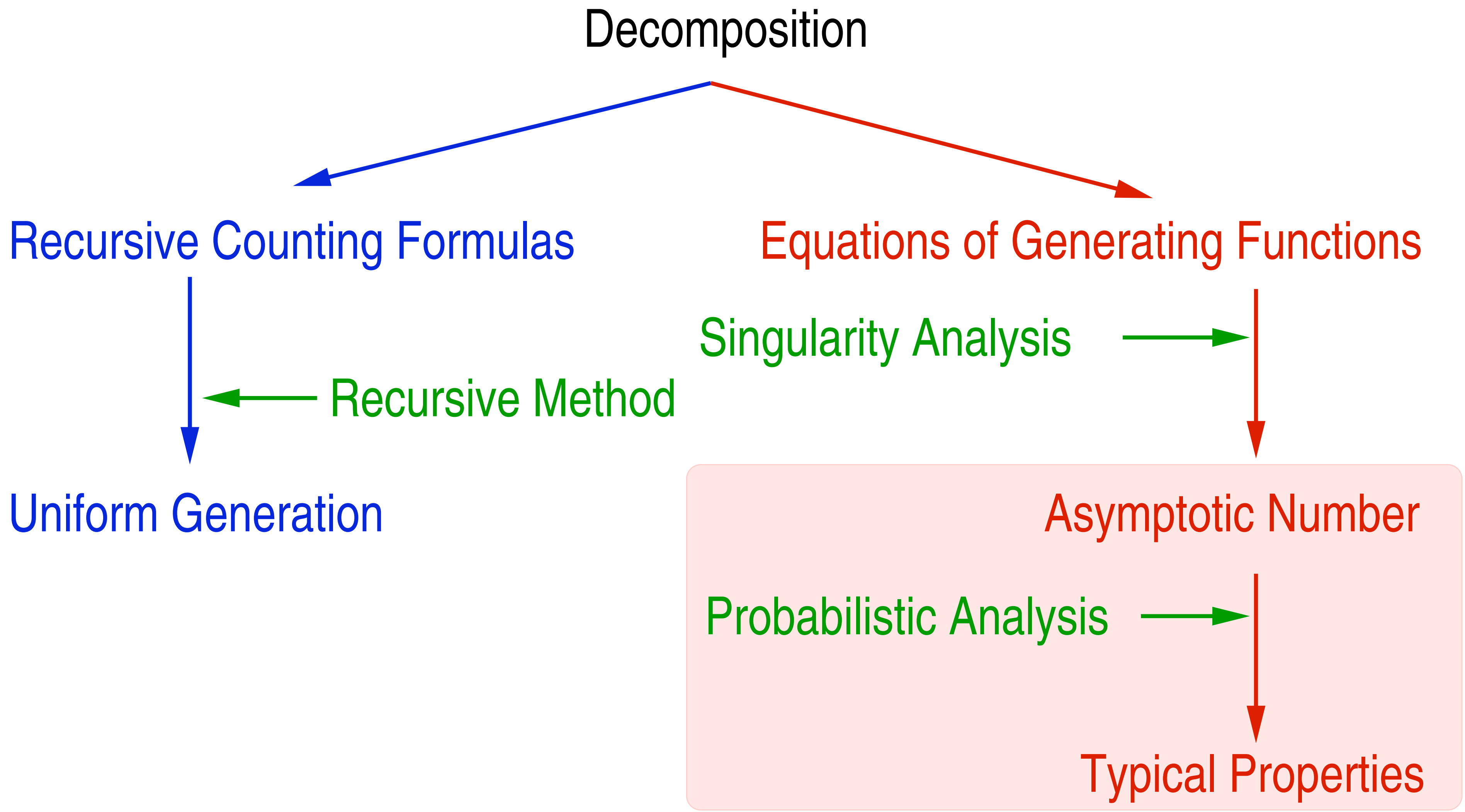
Scheme


Decomposition
Recursive Counting Formulas
Equations of Generating Functions
Singularity Analysis
Recursive Method
Uniform Generation
Asymptotic Number

Scheme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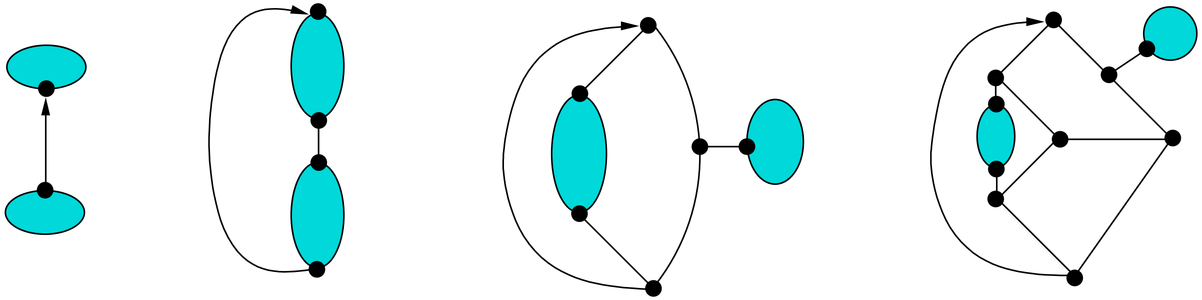
Typical Properties

Labeled cubic planar graphs

[Bodirsky, K., Löffler, McDiarmid 07]



The number of cubic planar graphs on n vertices is asymptotically

$$\sim \alpha n^{-7/2} \rho^n n! \,, \qquad \text{where} \;\; \rho \doteq 3.1325$$

Labeled cubic planar graphs

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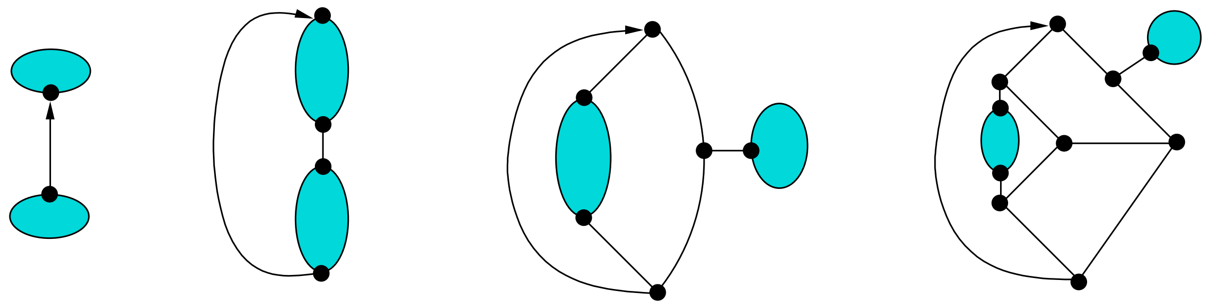

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What is the chromatic number of a random cubic planar graph G that is chosen uniformly at random among labeled cubic planar graphs on $[n]$?

Chromatic number

What is the chromatic number of a random cubic planar graph G?

- $\chi(G) \leq 4$ [Four colour theorem]
- For any connected graph G that is neither a complete graph nor an odd cycle, $\chi(G) \leq \Delta(G) = 3$ [Brooks' theorem]

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If G contains a component isomorphic to K_4, then $\chi(G) = 4$.

If G contains no isolated K_4, but at least one triangle, then $\chi(G) = 3$.

Random cubic planar graphs

[Bodirsky, K., Löffler, McDiarmid 07]

Let $G_n^{(k)}$ be a random k connected cubic planar graph on n vertices.

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Subgraph containments

Let X_n be $\#$ isolated K_4's in $G_n^{(0)}$ and Y_n $\#$ triangles in $G_n^{(k)}$, $k > 0$. Then

$$\lim_{n\to\infty} \Pr(X_n > 0) = 1 - e^{-\frac{\rho^4}{4!}}, \qquad \lim_{n\to\infty} \Pr(Y_n > 0) = 1.$$

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Chromatic number

$$\lim_{n\to\infty} \Pr(\chi(G_n^{(0)}) = 4) = \lim_{n\to\infty} \Pr(X_n > 0) = 1 - e^{-\frac{\rho^4}{4!}}$$

$$\lim_{n\to\infty} \Pr(\chi(G_n^{(0)}) = 3) = \lim_{n\to\infty} \Pr(X_n = 0, Y_n > 0) = e^{-\frac{\rho^4}{4!}} \doteq 0.9995\,.$$

For $k = 1, 2, 3,$ $\quad \lim_{n\to\infty} \Pr(\chi(G_n^{(k)}) = 3) = \lim_{n\to\infty} \Pr(Y_n > 0) = 1\,.$

Labeled planar structures

The number of planar structures on n vertices is asymp. $\sim \alpha\, n^{-\beta} \gamma^n n!$.

μn

k

Classes	β	γ				
Trees	$5/2$	2.71				
Outerplanar graphs	$5/2$	7.32				
Series-parallel graphs	$5/2$	9.07				
Planar graphs	$7/2$	27.2				
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Let G_n be a random planar structure on n vertices. Then as $n \to \infty$,

- the expected number of edges in G_n is $\sim \mu n$,
- G_n is connected with probability tending to a constant p_{con}, and
- $\chi(G_n)$ is three with probability tending to a constant p_χ.

k

Classes	β	γ	μ	p_{con}	p_χ	
Trees	$5/2$	2.71	1	1	0	
Outerplanar graphs	$5/2$	7.32	1.56	0.861	1	
Series-parallel graphs	$5/2$	9.07	1.61	0.889	?	
Planar graphs	$7/2$	27.2	2.21	0.963	?	
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Running time of uniform sampler (recursive method): $\tilde{O}(n^k)$

Classes	β	γ	μ	p_{con}	p_χ	k
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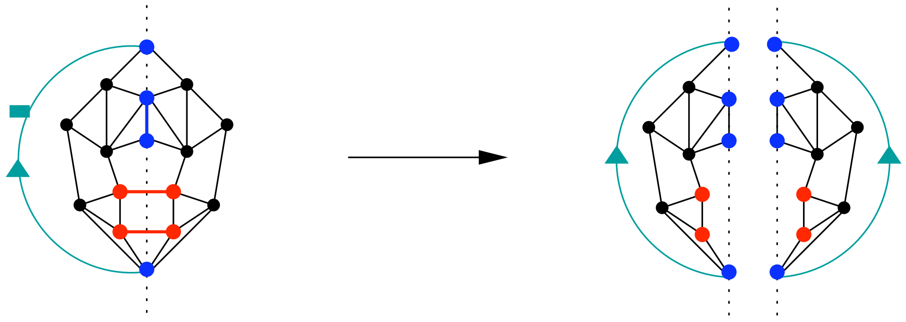

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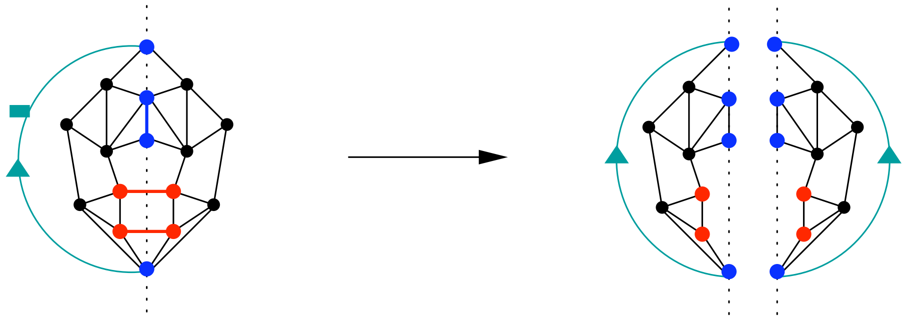

	Uniform sampling	Asymptotic number
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Planar graphs		

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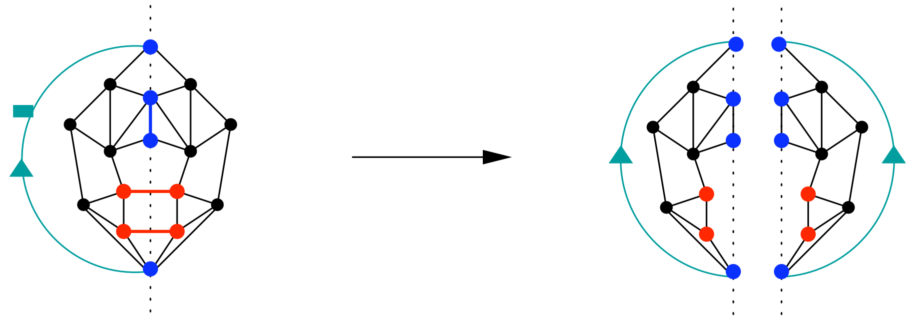

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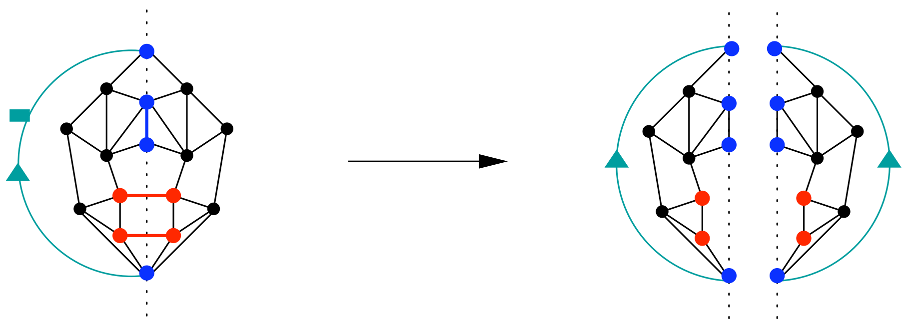

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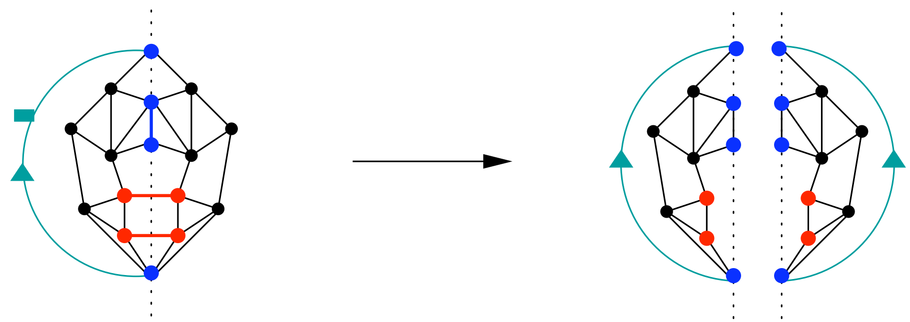

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Planar graphs	?	?

Outline

- Decomposition
- Recursive method
- Singularity analysis
- Probabilistic analysis

Outline

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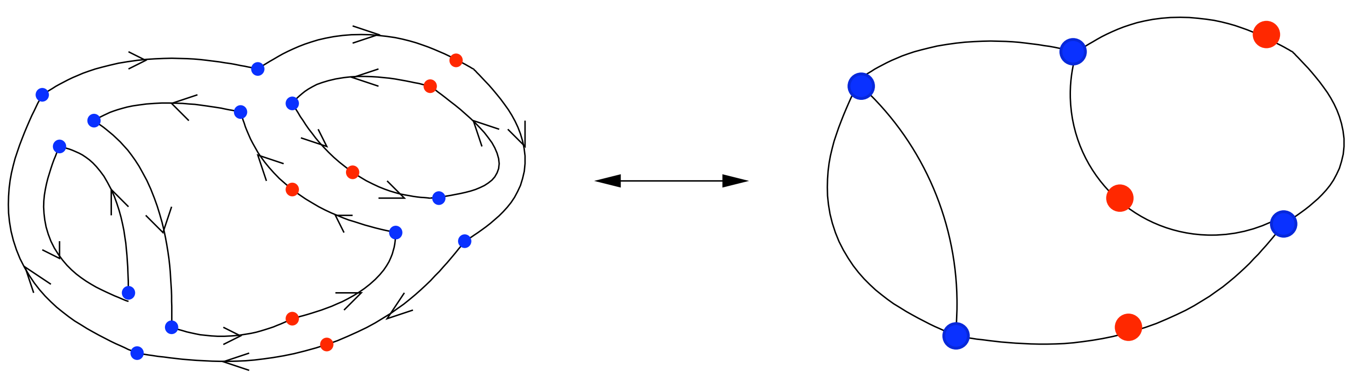
- Recursive method
- Singularity analysis
- Probabilistic analysis
- Gaussian matrix integral



Gaussian matrix integral

[WICK 50]

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The Gaussian matrix integral is defined by

$$< f > = \frac{\int f(M) e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM}{\int e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM},$$

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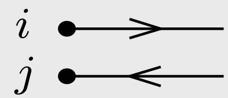
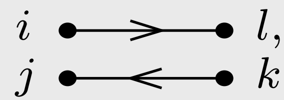
Pictorial interpretation

[Brézin, Itzykson, Parisi, Zuber 78; Zvonkin 97; di Francesco 04]

Pictorial interpretation from $< M_{ij} M_{kl} > = \frac{\delta_{il}\delta_{jk}}{N}$:

M_{ij} $\longleftrightarrow$ 

$< M_{ij} M_{kl} >= \frac{1}{N}$ $\longleftrightarrow$ 

$l = i$

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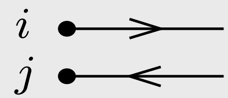

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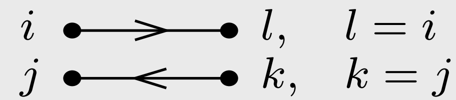

$$\mathrm{Tr}(M^n) \quad = \quad \sum_{1 \leq i_1, i_2, \cdots, i_n \leq N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}$$

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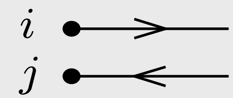

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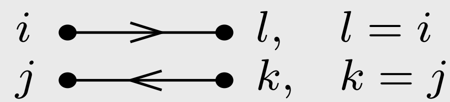

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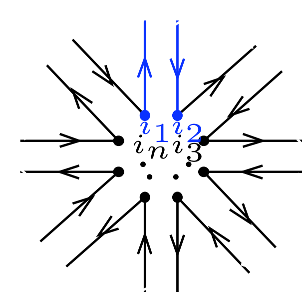

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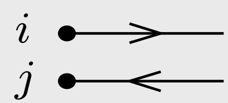

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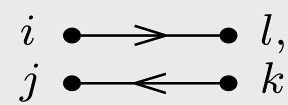

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$$= \sum_{1 \le i_1, i_2, \cdots, i_n \le N} \sum_{P} \prod_{(i_k i_{k+1}, i_l i_{l+1}) \in P} \frac{\delta_{i_k i_{k+1}} \delta_{i_l i_{l+1}}}{N}$$

where P is a partition of $\{i_1 i_2, i_2 i_3, \cdots, i_n i_1\}$ into pairs.

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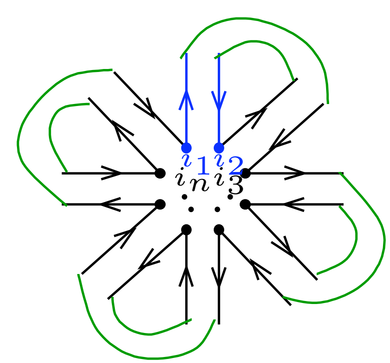

Fat graphs

[Brézin, Itzykson, Parisi, Zuber 78; Zvonkin 97; di Francesco 04]

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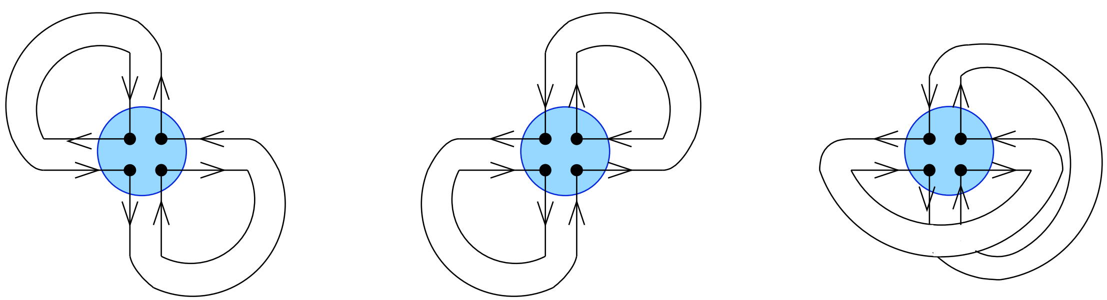
A pairing P with non-zero contribution to $< \mathrm{Tr}(M^n) >$

$\Longleftrightarrow$ a fat graph with one island and $n/2$ fat edges .



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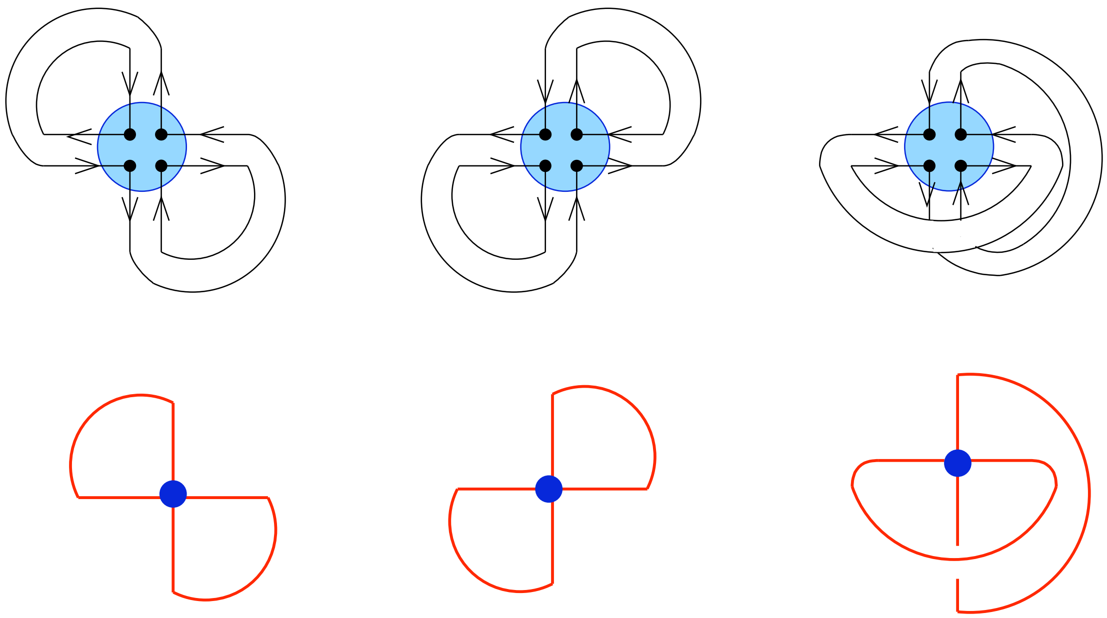
A pairing P with non-zero contribution to $< \mathrm{Tr}(M^n) >$

$\Longleftrightarrow$ a fat graph with one island and $n/2$ fat edges ordered cyclically.

(It defines uniquely an embedding on a surface: a map!)



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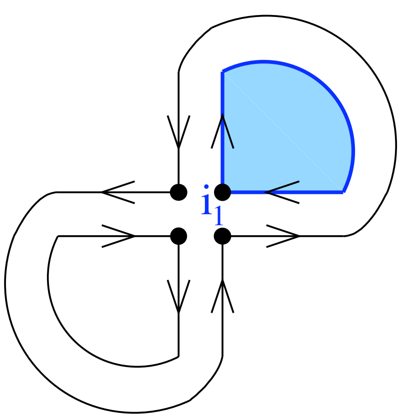
Let F be a fat graph with one island, $e(F)$ edges and $f(F)$ faces.

- The edges contribute $N^{-e(F)}$, since each edge contributes N^{-1}.
- The faces contribute $N^{f(F)}$, since each face attains independently any index from 1 to N.



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Thus

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where the sum is over all fat graphs F with one island.

Fat graphs

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Similarly we obtain

$$< \left[N\mathrm{Tr}(M^3)\right]^4 \left[N\mathrm{Tr}(M^2)\right]^3 > \quad = \quad \sum_F N^{7-e(F)+f(F)},$$

where the sum is over all fat graphs F with four islands of degree 3, and three islands of degree 2.

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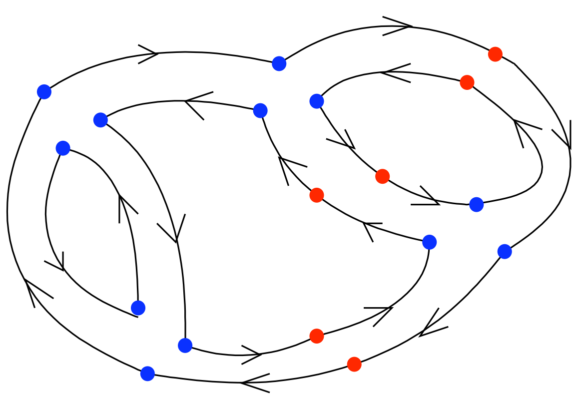

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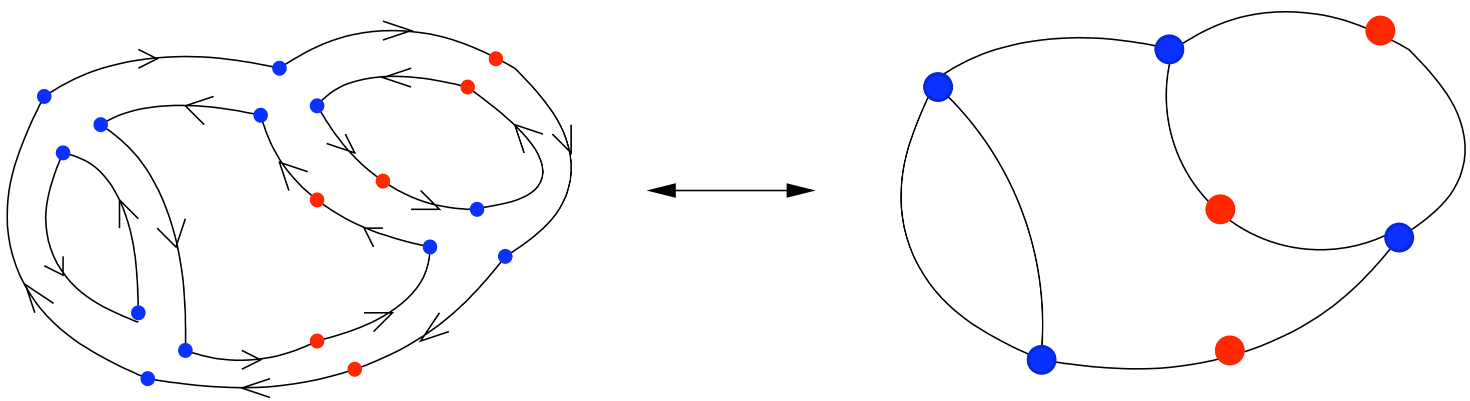

Planar maps

[Bouttier, Di Francesco, Guitter 02]

Let $g(M) = e^{\sum_{i\geq 1} \frac{z_i}{i}\left[N \mathrm{Tr}(M^i)\right]}$. Then

$$< g > = \sum_{a=(n_1,\cdots,n_k)} \sum_{F} N^{v(F)-e(F)+f(F)} \prod_{i\leq k} \frac{z_i^{n_i}}{i^{n_i} n_i!},$$

where F is a map with n_i vertices of degree i.

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$$\lim_{N \to \infty} \frac{\log < g >}{N^2} = \sum_{a=(n_1,\cdots,n_k)} \sum_{F_{cp}} \prod_{i \leq k} \frac{z_i^{n_i}}{i^{n_i} n_i!}$$

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[K., Loebl 06+]

The number of planar graphs with a given degree sequence can also be formulated by a Gaussian matrix intergral.

Concluding remarks

Relevant work

- There exists a constant c such that the number of graphs in a proper minor-closed class $\leq c^n \, n!$ [Norine, Seymour, Thomas, Wollan 06]

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Open problems

What are the asymptotic numbers of

(1) unlabeled planar graphs

(2) planar graphs with a given degree sequence

(3) embeddable graphs on a surface with higer genus?

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