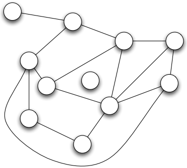
10 Steps to Counting Unlabeled Planar Graphs: 20 Years Later

Manuel Bodirsky

October 2007

A005470



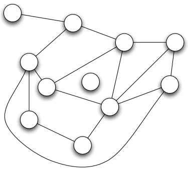
Sloane Sequence A005470 (core, nice, hard):

Number $p(n)$ of unlabeled planar simple graphs with n nodes.

Initial terms:

1, 2, 4, 11, 33, 142, 822, 6966, 79853, 1140916

A005470



Sloane Sequence A005470 (core, nice, hard):

Number $p(n)$ of unlabeled planar simple graphs with n nodes.

Initial terms:

1, 2, 4, 11, 33, 142, 822, 6966, 79853, 1140916

For comparison: number of all unlabeled graphs with n nodes

1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168

Unlabeled Enumeration

Consider graphs 'up to isomorphism'.

For general graphs: the number of labeled and the number of unlabeled graphs are asymptotically equal, since almost all graphs are asymetric.

For planar graphs: the number of labeled graphs is much larger than the number of unlabeled graphs, since almost all planar graphs have a large automorphisms group.

Tools for unlabeled enumeration:

1. ordinary generating functions
2. Burnside's lemma (orbit counting lemma)
3. cycle indices

Related Tasks

Problems related to the enumeration of planar graphs

- A Compute $p(n)$ in polynomial time in n.
- B Sample a random planar graph on n vertices in polynomial time in n from the uniform distribution.
- C Determine the asymptotic growth of $p(n)$.
- D Devise a Boltzman sampler for random planar graphs.
- E Analyse properties of random planar graphs

In our setting, all these tasks are closely related.

Liskovets and Walsh 1987: Ten Steps

V. A. Liskovets, T. R. Walsh: Ten steps to counting planar graphs, Congressus Numerantium (1987).

> *"One well-known long-standing unsolved graph-enumeration problem is to count (non-isomorphic) planar graphs. The aim of this brief survey is to draw the reader's attention to the considerable progress which has been achieved to that end, and which suggests that this problem may soon be completely solved."*

Liskovets and Walsh 1987: Ten Steps

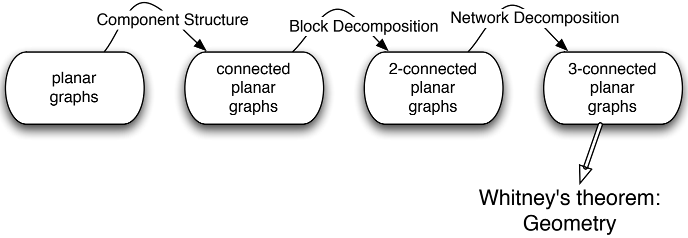
V. A. Liskovets, T. R. Walsh: Ten steps to counting planar graphs, Congressus Numerantium (1987).

> *"One well-known long-standing unsolved graph-enumeration problem is to count (non-isomorphic) planar graphs. The aim of this brief survey is to draw the reader's attention to the considerable progress which has been achieved to that end, and which suggests that this problem may soon be completely solved."*

The problem is still open.

General Approach

Essentially, there is no alternative to the following basic approach:



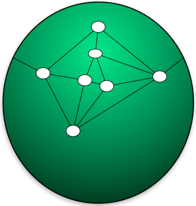
For the labeled case, this approach has been successful:

- for exact numbers and random generation (B.,Gröpl,Kang'03)
- for the asymptotic growth (Gimenez,Noy'05)
- for Boltzmann generation (Fusy'05)

Planar Maps

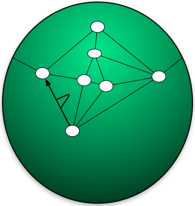
How to count 3-connected planar maps, i.e., 3-connected plane graphs on the sphere, up to homeomorphisms?

A question that is credited to Euler $\longrightarrow$ polyhedra



Planar Maps

How to count 3-connected planar maps, i.e., 3-connected plane graphs on the sphere, up to homeomorphisms?
A question that is credited to Euler $\longrightarrow$ polyhedra

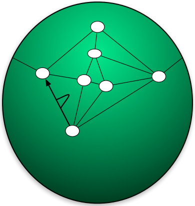


Tutte:

- First count *rooted maps*,
  (and try to get rid of the root later...)

Planar Maps

How to count 3-connected planar maps, i.e., 3-connected plane graphs on the sphere, up to homeomorphisms?
A question that is credited to Euler $\longrightarrow$ polyhedra



Tutte:

- First count *rooted maps*,
  (and try to get rid of the root later...)
- Go from lower connectivity to higher connectivity

Ten Steps

symmetry type	maps	non-sep. maps	3-conn. maps
rooted	Tutte'63	Tutte'63	Tutte'63
sense-pres. iso.	Liskovets'82	Liskovets,Walsh'83	Walsh'82
all map-iso.	Wormald'81	Wormald'xx	Wormald'xx

Ten Steps

symmetry type	maps	non-sep. maps	3-conn. maps
rooted	Tutte'63	Tutte'63	Tutte'63
sense-pres. iso.	Liskovets'82	Liskovets,Walsh'83	Walsh'82
all map-iso.	Wormald'81	Wormald'xx	Wormald'xx

Last step from 3-connected to planar graphs: 'recursive scheme'

Ten Steps

symmetry type	maps	non-sep. maps	3-conn. maps
rooted	Tutte'63	Tutte'63	Tutte'63
sense-pres. iso.	Liskovets'82	Liskovets,Walsh'83	Walsh'82 Fusy'05
all map-iso.	Wormald'81	Wormald'xx	Wormald'xx

Last step from 3-connected to planar graphs: 'recursive scheme'

Ten Steps

symmetry type	maps	non-sep. maps	3-conn. maps
rooted	Tutte'63	Tutte'63	Tutte'63
sense-pres. iso.	Liskovets'82	Liskovets,Walsh'83	Walsh'82 Fusy'05
all map-iso.	Wormald'81	Wormald'xx	Wormald'xx

Last step from 3-connected to planar graphs: 'recursive scheme'

Cubic Planar Graphs:

symmetry type	maps	non-sep. maps	3-conn. maps
rooted	Mullin'66	Mullin'65	Tutte'64
sense-pres. iso.	Liskovets,Walsh'87	(Brown'64)	
all map-iso.			Tutte'80

The Orbit-counting Lemma

Aka Cauchy-Frobenius, or Burnside Lemma.
G finite group acting on a set X. The number of orbits of G is

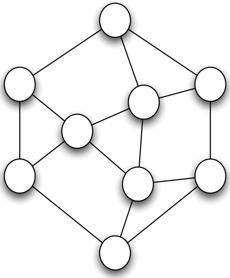
$$1/|G| \sum_{g \in G} |Fix(g)|$$

In our setting, for 3-connected planar graphs:

- X: 3-connected labeled planar graphs with vertices $\{1, \ldots, n\}$
- $G = \{$id,reflections,rotations,reflection-rotations$\}$
- Orbits of G: unlabeled 3-connected planar graphs

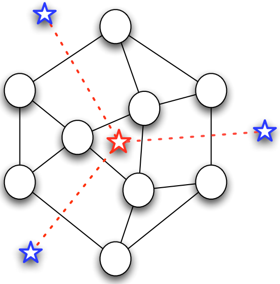
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



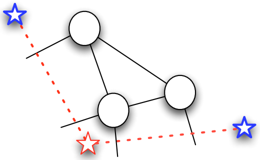
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



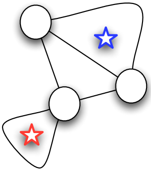
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



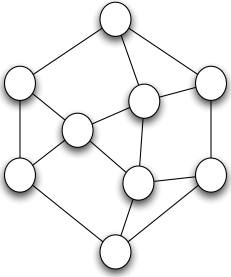
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



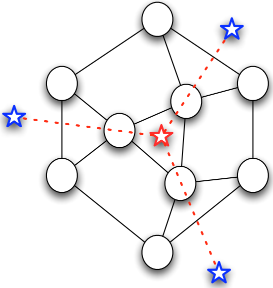
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



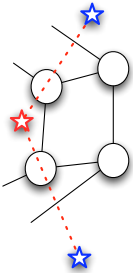
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



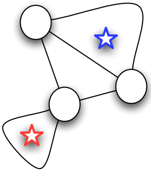
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:



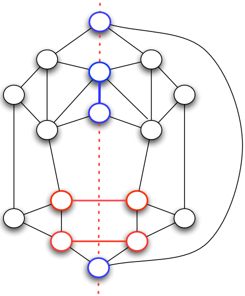
Quotient Maps

To count graphs with a rotative symmetry of order k, use concept of quotient maps (Liskovets'82,Walsh'82,Fusy'05).
Example for $k = 3$ where both poles of the rotation are faces:

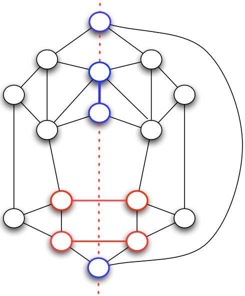


Obtain a unique map with two distinguished faces
Can be further decomposed
(e.g. by using quadrangulations as in Fusy'04)

Maps with a Reflective Symmetry



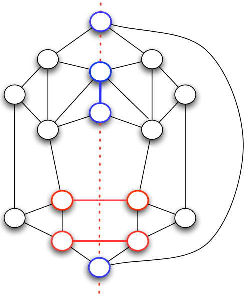
Maps with a Reflective Symmetry



Several Decompositions and Algorithms:

- Wormald'xx (unpublished algorithm)

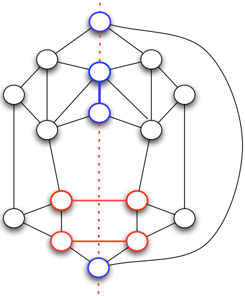
Maps with a Reflective Symmetry



Several Decompositions and Algorithms:

- Wormald'xx (unpublished algorithm)
- Qadrangulation method in Fusy'05 can in principle be applied here as well

Maps with a Reflective Symmetry



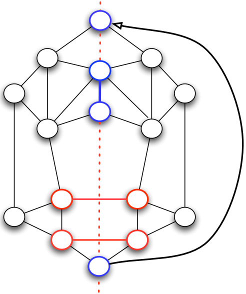
Several Decompositions and Algorithms:

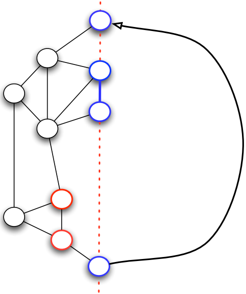
- Wormald'xx (unpublished algorithm)
- Qadrangulation method in Fusy'05 can in principle be applied here as well
- B.,Groepl,Kang'05: Colored connectivity decomposition

None of the approaches lead to reasonable formulas so far

Colored Decomposition

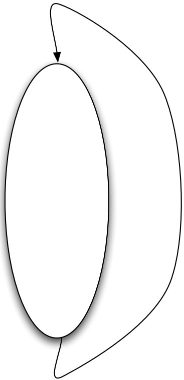
Assume that there is a distinguished directed edge on the symmetry (an *arc-root*).





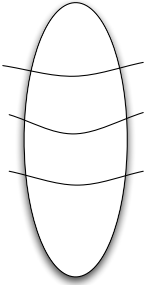
Resulting graph is 2-connected, and can be decomposed easily.
But: have two parameters for number of red and blue vertices

Tutte-like Decomposition



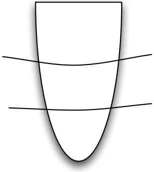
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



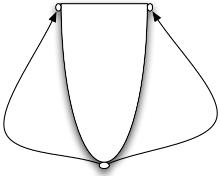
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



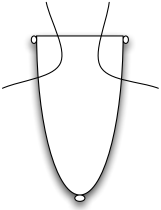
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



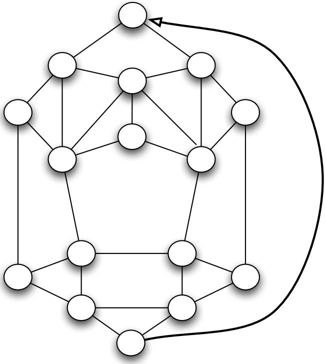
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



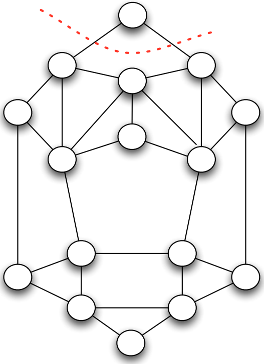
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



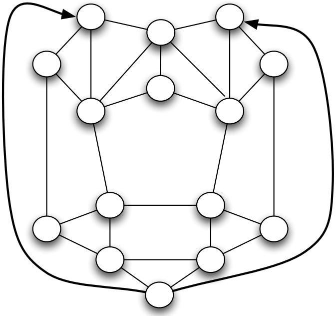
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



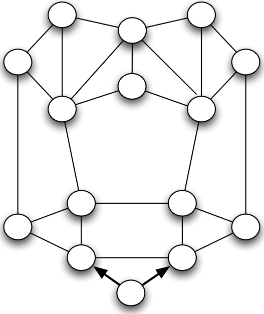
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



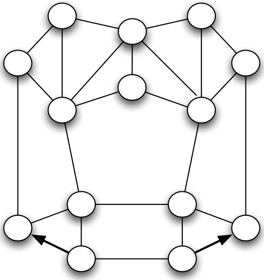
Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)

Tutte-like Decomposition



Similarly to the decomposition of triangulations (Tutte')
and c-nets (B.,Groepl,Johannsen,Kang'05)
Advantage: only one extra variable, simple GF equations.
But: tedious

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.
Forest	K_3	No 2-conn. comp.

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.
Forest	K_3	No 2-conn. comp.

Results:

Graph Class	Labeled	Unlabeled
Planar		
Series-parallel		
Outerplanar		
Forest	Well-known	Well-known

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.
Forest	K_3	No 2-conn. comp.

Results:

Graph Class	Labeled	Unlabeled
Planar	Gimenez,Noy'05	
Series-parallel		
Outerplanar		
Forest	Well-known	Well-known

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.
Forest	K_3	No 2-conn. comp.

Results:

Graph Class	Labeled	Unlabeled
Planar	Gimenez,Noy'05	
Series-parallel	B.,Kang,Gimenez,Noy'05	
Outerplanar	B.,Kang,Gimenez,Noy'05	
Forest	Well-known	Well-known

Another Line of Research

Can make progress already before we solve Euler's problem.

Graph Class	Forbidden Minors	Connectivity Structure
Planar	$K_5, K_{3,3}$	Whitney for 3-conn.
Series-parallel	K_4	No 3-conn. comp.
Outerplanar	$K_4, K_{2,3}$	Hamiltonian 2-conn comp.
Forest	K_3	No 2-conn. comp.

Results:

Graph Class	Labeled	Unlabeled
Planar	Gimenez,Noy'05	?
Series-parallel	B.,Kang,Gimenez,Noy'05	?
Outerplanar	B.,Kang,Gimenez,Noy'05	B.,Fusy,Kang,Vigerske'07
Forest	Well-known	Well-known

Cycle Indices

Polya theory.
Let G be a graph with vertices $\{1, \ldots, n\}$.

$$Z(G; s_1, s_2, \ldots) := 1/|Aut(G)| \sum_{g \in Aut(G)} \prod_{k=1}^{n} s_k^{j_k}(g)$$

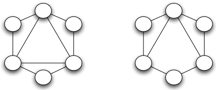
where $j_k(g)$ is the number of cycles of length k in g.

Let $\mathcal{K}$ be a class of graphs.

$$Z(\mathcal{K}; s_1, s_2, \ldots) := \sum_{G \in \mathcal{K}} Z(G; s_1, s_2, \ldots)$$

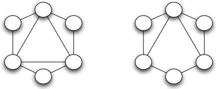
2-Connected Outerplanar Graphs

Cycle index sum for 2-connected outerplanar graphs



2-Connected Outerplanar Graphs

Cycle index sum for 2-connected outerplanar graphs

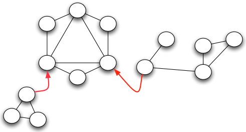


$$Z(\mathcal{D}) = -\frac{1}{2}\sum_{d\geq 1}\frac{\varphi(d)}{d}\log\left(\frac{3}{4}-\frac{1}{4}s_d+\frac{1}{4}\sqrt{s_d^2-6s_d+1}\right) + \frac{s_2+s_1^2-4s_1-2}{16}+\frac{s_1^2-3s_1^2s_2+2s_1s_2}{16s_2^2}+\frac{3-s_1}{16}\sqrt{s_1^2-6s_1+1} - \frac{1}{16}\left(1+\frac{s_1^2}{s_2^2}+\frac{2s_1}{s_2}\right)\sqrt{s_2^2-6s_2+1},$$

where φ is the Euler-φ-function $\varphi(n) = n\prod_{p|n}(1-p^{-1})$

From 2-Connected to Connected

Technique be Norman'54, Robinson'70, Harary,Palmer'73



Tool 1: composition

$$Z(G)[Z(\mathcal{K})] := Z(G; Z(\mathcal{K}; s_1, s_2, \dots), Z(\mathcal{K}; s_2, s_4, \dots), \dots)$$

Tool 2: rooting

$$Z(\hat{\mathcal{G}}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{G})$$

Tool 3: unrooting

$$Z(\mathcal{G}) = \int_0^{s_1} \frac{1}{t_1} Z(\hat{\mathcal{G}})|_{s_1 = t_1} dt_1 + Z(\mathcal{G})|_{s_1=0}$$

From 2-Connected to Connected

$\hat{\mathcal{D}}$: cycle index sum for rooted two-connected outerplanar graphs.
$\hat{\mathcal{C}}$: cycle index sum for rooted connected outerplanar graphs.

$$Z(\hat{\mathcal{C}}) = s_1 \exp\left(\sum_{k\geq 1} \frac{Z\left(\hat{\mathcal{D}}; Z\left(\hat{\mathcal{C}}; s_k, s_{2k}, \ldots\right), Z\left(\hat{\mathcal{C}}; s_{2k}, s_{4k}, \ldots\right)\right)}{k\, Z\left(\hat{\mathcal{C}}; s_k, s_{2k}, \ldots\right)}\right)$$

The cycle index sum for connected outerplanar graphs

$$Z(\mathcal{C}) = Z(\hat{\mathcal{C}}) + Z\left(\mathcal{D}; Z(\hat{\mathcal{C}})\right) - Z\left(\hat{\mathcal{D}}; Z(\hat{\mathcal{C}})\right)$$

Substituting x^i for s_i gives equations for the generating functions and a polynomial-time algorithm to compute the numbers.

Asymptotic Results

With singularity analysis (Flajolet,Sedgewick'0x) we get

Theorem 1 (B.,Fusy,Kang,Vigerske'07).

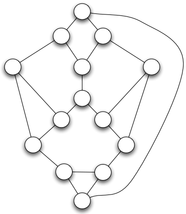
The numbers d_n, c_n, and g_n of two-connected, connected, and general outerplanar graphs with n vertices have the asymptotic estimates

$$\begin{aligned} d_n &\sim d\, n^{-5/2}\delta^{-n}, \\ c_n &\sim c\, n^{-5/2}\rho^{-n}, \\ g_n &\sim g\, n^{-5/2}\rho^{-n}, \end{aligned}$$

with exponential growth rates $\delta^{-1} = 3 + 2\sqrt{2} \approx 5.82843$ and $\rho^{-1} \approx 7.50360$, and constants $d \approx 0.00596026$, $c \approx 0.00760471$, and $g \approx 0.00909941$.

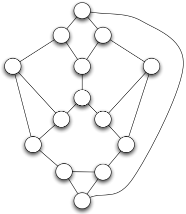
Cubic Planar Graphs

All vertices of degree three.



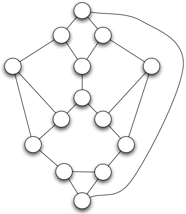
Cubic Planar Graphs

All vertices of degree three.
Analysed for the labeled case in
B.,McDiarmid,Loeffler,Kang'07.



Cubic Planar Graphs

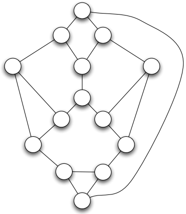
All vertices of degree three.
Analysed for the labeled case in B.,McDiarmid,Loeffler,Kang'07.



Why interesting?

Cubic Planar Graphs

All vertices of degree three.
Analysed for the labeled case in
B.,McDiarmid,Loeffler,Kang'07.

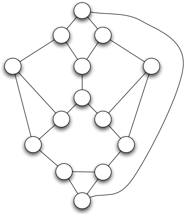


Why interesting?

- More difficult than SP graphs in that we have to deal with 3-connected components

Cubic Planar Graphs

All vertices of degree three.
Analysed for the labeled case in
B.,McDiarmid,Loeffler,Kang'07.

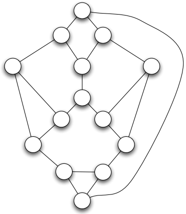


Why interesting?

- More difficult than SP graphs in that we have to deal with 3-connected components
- However, the number of 3-connected cubic planar graphs is well-understood (bijective correspondence to triangulations)

Cubic Planar Graphs

All vertices of degree three.
Analysed for the labeled case in
B.,McDiarmid,Loeffler,Kang'07.



Why interesting?

- More difficult than SP graphs in that we have to deal with 3-connected components
- However, the number of 3-connected cubic planar graphs is well-understood (bijective correspondence to triangulations)
- 2-connected and connected numbers are closely related

Ten Steps

Ten Steps

1. 2-connected, connected, and general series-parallel graphs

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation
6. Polyhedra (3-connected planar graphs)

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation
6. Polyhedra (3-connected planar graphs)
7. Arc-rooted 2-connected planar graphs

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation
6. Polyhedra (3-connected planar graphs)
7. Arc-rooted 2-connected planar graphs
8. 2-connected planar graphs

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation
6. Polyhedra (3-connected planar graphs)
7. Arc-rooted 2-connected planar graphs
8. 2-connected planar graphs
9. Connected planar graphs

Ten Steps

1. 2-connected, connected, and general series-parallel graphs
2. 2-connected, connected, and general cubic planar graphs
3. Arc-rooted 3-connected planar maps with a sense-reversing automorphism
4. 3-connected planar graphs with a sense-reversing automorphism
5. 3-connected planar graphs with a sense-reversing rotation
6. Polyhedra (3-connected planar graphs)
7. Arc-rooted 2-connected planar graphs
8. 2-connected planar graphs
9. Connected planar graphs
10. Planar graphs