Cycle Index Series Factoring in Enumerative Group Theory

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Overview of the Results

Problem statement

We propose a classification of the subgroups, and their conjugacy classes, in a free product of cyclic groups like $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$.

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- Etc..

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- A new family of CAT algorithms to generate the examples.
- A fascinating connection with *Combinatorial Maps*.

For the sake of simplicity, we shall focus on the *modular group* example.

Part I

Beautiful Structures

Definition 1

Trivalent Diagrams are bicolored $\{\bullet, \circ\}$ connected graphs with cyclic orientation and degree conditions at the vertices.

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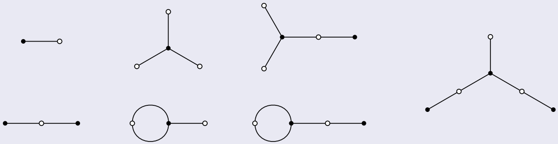
- Trivalent black vertices are cyclically oriented.

Definition 2

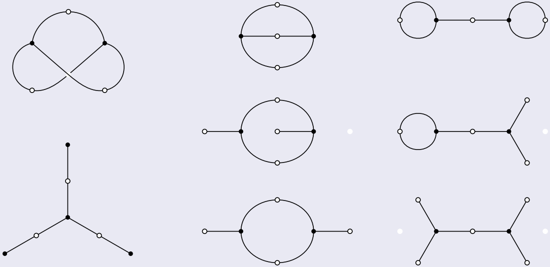
A *morphism* φ between two trivalent diagrams is a collection of three maps $\varphi_\bullet$, $\varphi_\circ$ and φ_- sending the *black vertices*, *white vertices*, and the *edges* of the first diagram to corresponding elements of the second diagram, preserving *adjacencies* and *cyclic orientations*.

Beautiful Structures

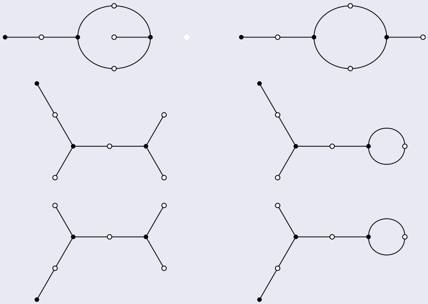
Examples : trivalent diagrams of size up to *five*.



Examples : trivalent diagrams of size *six*.



Examples : trivalent diagrams of size *seven*.



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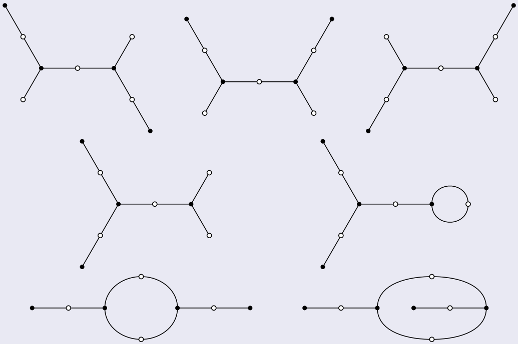
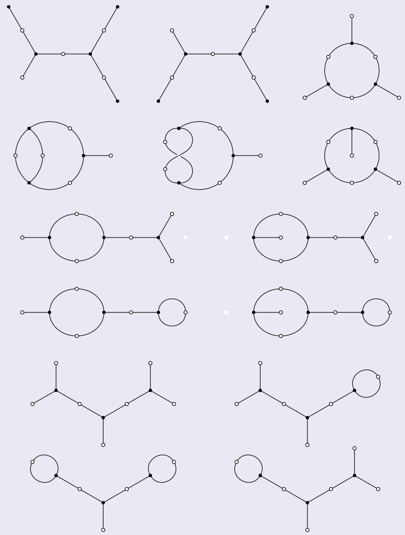
Examples : trivalent diagrams of size *eight*.



Beautiful Structures

Examples : trivalent diagrams of size *nine*.



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Pointed Trivalent Diagrams are trivalent diagrams with a distinguished edge, which is called the *base point* of the diagram (*sic*).

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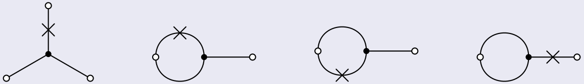
Examples : pointed trivalent diagrams of size *three*.



Part II

A Fully Explicit and Computable Classification

- The *conjugacy class of subgroups* of the *modular group* are classified by *trivalent diagrams*.

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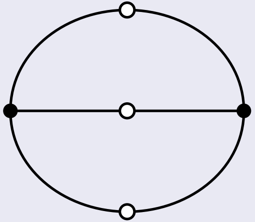
- The subgroup *themselves* are classified by *pointed* trivalent diagrams.

Recall

The Modular Groups of level n, are the subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ consisting of matrix (*up to sign*) congruent to the identity modulo n. We have the following short exact sequence.

$$0 \longrightarrow \Gamma_n \longrightarrow \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

$n = 2$: the *triangle*.



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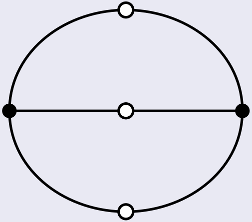
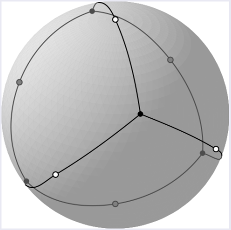


$n = 3$: the *tetrahedron*.



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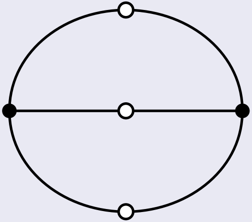

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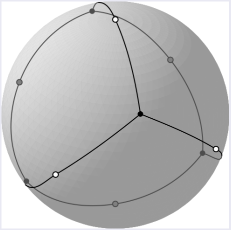

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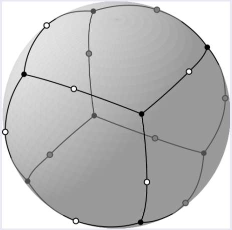

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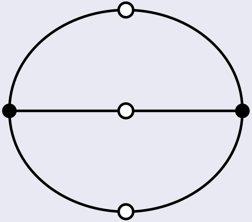

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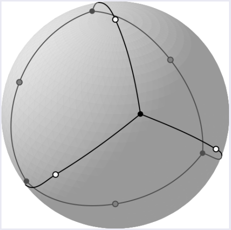

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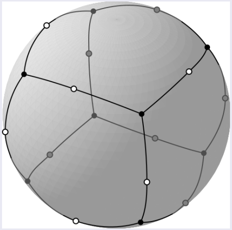
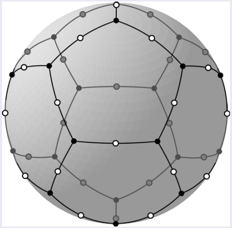
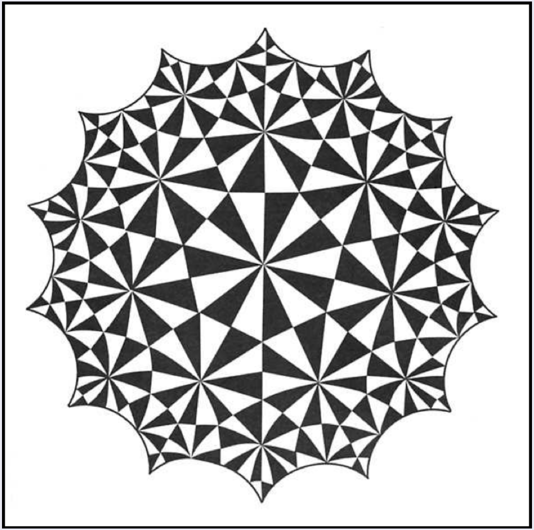


$n = 5$: the *dodecahedron*.



$n = 7$: Klein's cubic.



Part III

Counting Principles

Two isomorphisms

- The species D_3^* of *not necessarily connected* trivalent diagrams is isomorphic to the *direct product* of the species S_2 and S_3 of permutations of compositional order *two* and *three*.

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- The species D_3 of *connected* trivalent diagrams is related to D_3^* by the following isomorphism.

$$D_3^* \underset{nat.}{\simeq} \mathrm{Set}(D_3)$$

Expressing the *existence* and *uniqueness* of the decomposition of a diagram in its *connected components*.

Counting Principles

Let's introduce the related generating series

	Pointed	Not pointed
Labeled	$D_3^\bullet(t) = \sum_{n\geq 0} \frac{a_n^\bullet}{n!} t^n$	$D_3(t) = \sum_{n\geq 0} \frac{a_n}{n!} t^n$
Unlabeled	$\tilde{D}_3^\bullet(t) = \sum_{n\geq 0} \tilde{a}_n^\bullet t^n$	$\tilde{D}_3(t) = \sum_{n\geq 0} \tilde{a}_n t^n$

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We deduce the following equations

- On Hurwitz generating series,

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- On cycle index series,

$$\mathcal{Z}_{D_3^*}(x_1, x_2, \dots) = \mathcal{Z}_{S_2}(x_1, x_2, \dots) \odot \mathcal{Z}_{S_3}(x_1, x_2, \dots) \quad \text{and}$$
$$\mathcal{Z}_{D_3}(x_1, x_2, \dots) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \mathcal{Z}_{D_3^*}(x_k, x_{2k}, \dots)$$

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Enabling this way, an *unlabeled* counting from a *labeled* one.

The number of *pointed trivalent diagrams*. (A005133)

$$\begin{aligned}\tilde{D}_3^{\bullet}(t) &= t + t^2 + 4\,t^3 + 8\,t^4 + 5\,t^5 + 22\,t^6 + 42\,t^7 + 40\,t^8 \\ &+ 120\,t^9 + 265\,t^{10} + 286\,t^{11} + 764\,t^{12} + 1729\,t^{13} \\ &+ 2198\,t^{14} + 5168\,t^{15} + 12144\,t^{16} + 17034\,t^{17} \\ &+ 37702\,t^{18} + 88958\,t^{19} + 136584\,t^{20} + 288270\,t^{21} \\ &+ 682572\,t^{22} + 1118996\,t^{23} + 2306464\,t^{24} \\ &+ 5428800\,t^{25} + 9409517\,t^{26} + 19103988\,t^{27} \\ &+ 44701696\,t^{28} + 80904113\,t^{29} + 163344502\,t^{30} + \dots\end{aligned}$$

Part IV

A New Way to Compute Cycle Index Series

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Recall

A *generic* cycle index series, or Joyal-Pólya series, has the following form.

$$\mathcal{Z}(x_1, x_2, \ldots) = \sum_{n \geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{a_{k_1,\ldots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!} x_1^{k_1} \cdots x_n^{k_n}$$

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A remark on *complexity*

The general series contains exactly p_n terms in its n-th *graduation* and $p_1 + p_2 + \cdots + p_n$ terms in its n-th *filtration*.

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e.g. more than a million terms in the 50-th filtration.

A New Way to Compute Cycle Index Series

Lemma

Let $\mathcal{Z}_F$ and $\mathcal{Z}_G$ be the cycle index series of two combinatorial species F and G,

$$\mathcal{Z}_F = \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{a_{k_1,\ldots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n} \text{ et}$$

$$\mathcal{Z}_G = \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_{k_1,\ldots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n}$$

Then, the cycle index series of their direct product $F \times G$ is simply the *Hadamard product* of those two series,

$$\mathcal{Z}_F \odot \mathcal{Z}_G \overset{\text{def.}}{=} \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{a_{k_1,\ldots,k_n}\, b_{k_1,\ldots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n}$$

Definition

A Joyal-Pólya series is said to be *separated* if it admits an expression of the following form,

$$\mathcal{Z}(x_1, x_2, \dots) = \prod_{k \geq 1} \left(\sum_{n \geq 0} \frac{a_{k,n}}{k^n \, n!} \, x_k^n \right) \qquad \text{with } a_{k,0} = 1 \text{ for all } k \geq 1.$$

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A remark on *complexity*

The general series in the *factored form* is very *sparse*. It contains $n + n/2 + n/3 + \cdots = O(n \, \log n)$ terms in its n-th *filtration*.

Lemma

Let,

$$\mathcal{Z}_1 = \prod_{k\geq 1} \left(\sum_{n\geq 0} \frac{a_{k,n}}{k^n\, n!}\, x_k^n \right) \quad \text{et} \quad \mathcal{Z}_2 = \prod_{k\geq 1} \left(\sum_{n\geq 0} \frac{b_{k,n}}{k^n\, n!}\, x_k^n \right)$$

are two cycle index series in *factored* form, then,

$$\mathcal{Z}_1 \odot \mathcal{Z}_2 = \prod_{k\geq 1} \left(\sum_{n\geq 0} \frac{a_{k,n}\, b_{k,n}}{k^n\, n!}\, x_k^n \right)$$

The number of *trivalent diagrams*. (A121350)

$$\begin{aligned}\tilde{D}_3(t) \;=\;& t + t^2 + 2\,t^3 + 2\,t^4 + t^5 + 8\,t^6 + 6\,t^7 + 7\,t^8 + 14\,t^9 \\ &+ 27\,t^{10} + 26\,t^{11} + 80\,t^{12} + 133\,t^{13} + 170\,t^{14} + 348\,t^{15} \\ &+ 765\,t^{16} + 1002\,t^{17} + 2176\,t^{18} + 4682\,t^{19} + 6931\,t^{20} \\ &+ 13740\,t^{21} + 31085\,t^{22} + 48652\,t^{23} + 96682\,t^{24} \\ &+ 217152\,t^{25} + 362779\,t^{26} + 707590\,t^{27} + 1597130\,t^{28} \\ &+ 2789797\,t^{29} + 5449439\,t^{30} + \ldots\end{aligned}$$

Part V

Combinatorial Maps

Problem Statement

Topologically, a *combinatorial map* on a closed orientable surface S is a graph regularly embedded in S such that its complementary is made of polygonal regions, the *faces* of the map.

The *genus* g of the map is the genus of the underlying surface. It can be computed from the Euler-Poincaré characteristic of the map,

$$\chi_E = n_v - n_e + n_f \qquad\qquad g = 1 - \frac{1}{2}\chi_E$$

An *isomorphism* of map is an orientation preserving diffeomorphism of the underlying surfaces sending vertices on vertices, edges on edges, etc...

Problem Statement

Combinatorial maps can be described up to isomorphism by various combinatorial invariants. For example triples of permutation σ_v, σ_e and σ_f on the set of directed edges of the map, with $\sigma_f\,\sigma_e\,\sigma_v = 1$, σ_a of order two ($\sigma_a^2 = 1$) and having no fixed point.

The fundamental enumeration problem in combinatorial map theory is then to count the number of combinatorial maps up to isomorphism according to various combination of the following parameters, the number of its vertices n_v, the number of its edges n_e, the number of its faces n_f, and its genus g.

For now, only partial results are known. Every known result falls in two sort of restrictions :

1. Supplements of structure : coloring, pointing, labeling, assigning a fixed value to some of the parameters, etc...
2. Defects of structure : letting some of the parameters unspecified, getting ride of some of the constraints, etc...

Results falling in the second class are in general much more difficult because of the presence of *symmetries*. In contrast, the considered supplements of structure kill all the symmetries and counting *rigid* structures is much easier.

The most precise results in the *first* class known up to now are contained in,

1. W. T. TUTTE, *On the enumeration of planar maps*, Bull. Amer. Math. Soc. 74 (1968), 64-74.
2. T.R.S. WALSH, A.B. LEHMAN, *Counting rooted maps by genus I*, J. Combin. Theory Ser. B 13 (1972), 192-218.
3. T. R. S. WALSH, A. B. LEHMAN, *Counting rooted maps by genus II*, J. Combin. Theory Ser. B 14 (1973), 122-141.
4. V. A. LISKOVETS, *Enumeration of nonisomorphic planar maps*, Selecta Math. Sovietica 4 (1985) 303-323.

State of the Art

The most precise results in the *first* class known up to now are contained in, (*continued*)

5. E. A. BENDER, E.A. CANFIELD, R.W. ROBINSON, *The enumeration of maps on the torus and on the projective plane*, Canad. Math. Bull. 31 (1988) 257-271.
6. E. A. BENDER, E.A. CANFIELD, *The number of rooted maps on an orientable surface*, J. Combin. Theory Ser. B 53 (1991) 293-299.
7. M. BOUSQUET, G. LABELLE, P. LEROUX, *Enumeration of planar two-face maps*, Discrete Math. 222 (2000), 1-25.

State of the Art

The most precise results in the *second* class known up to now are contained in,

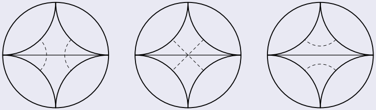
1. T. R. S. WALSH, *Generating nonisomorphic maps without storing them*, SIAM J. Algebraic Discrete methods 4 (1983), 161-178.
2. A. MEDNYKH, R. NEDELA, *Enumeration of unrooted maps with given genus*. J. Combin. Theory Ser. B 96 (Sept. 2006), 706-729.

Work in Progress by the lecturer ;-)

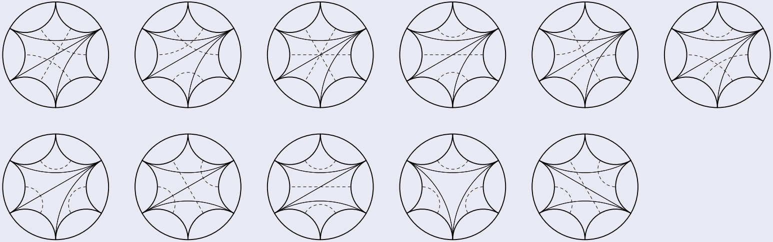
1. Compare the output of the new generating algorithms with that of (T. R. S. WALSH 1983).
2. Compute the number of combinatorial maps up to isomorphism by genus, number of vertices, edges and faces, with the help of a new counting principle (*tiresome computation...*)
3. Compare that result to that of (A. MEDNYKH, R. NEDELA 2006).
4. Refine that result with the *degree list* of vertex and faces instead of just their numbers.

Samples of Results

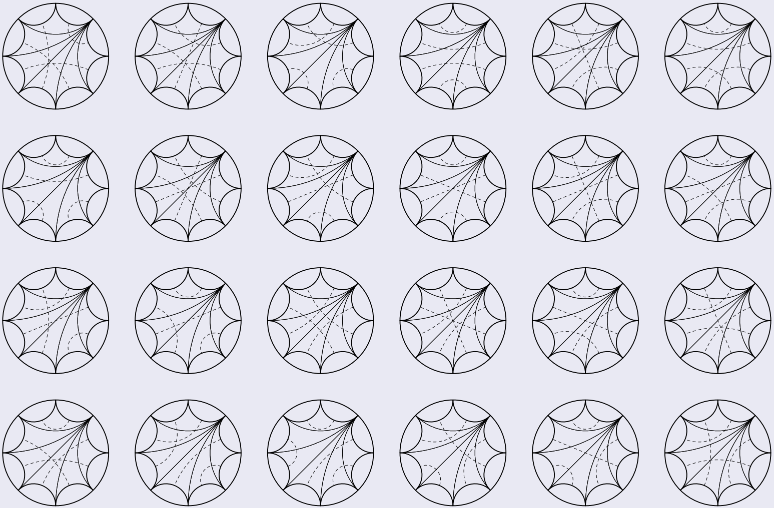
The *three* triangular maps with *two* faces



The *eleven* triangular maps with *four* faces

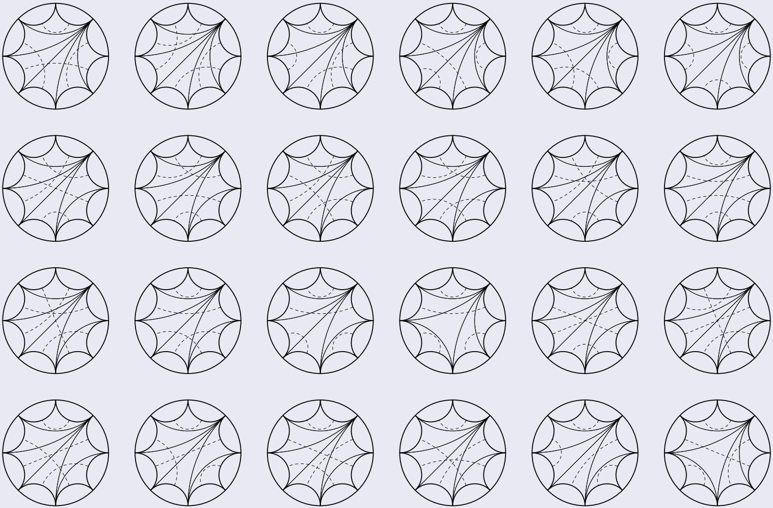


The *eighty one* triangular maps with *six* faces (first part)



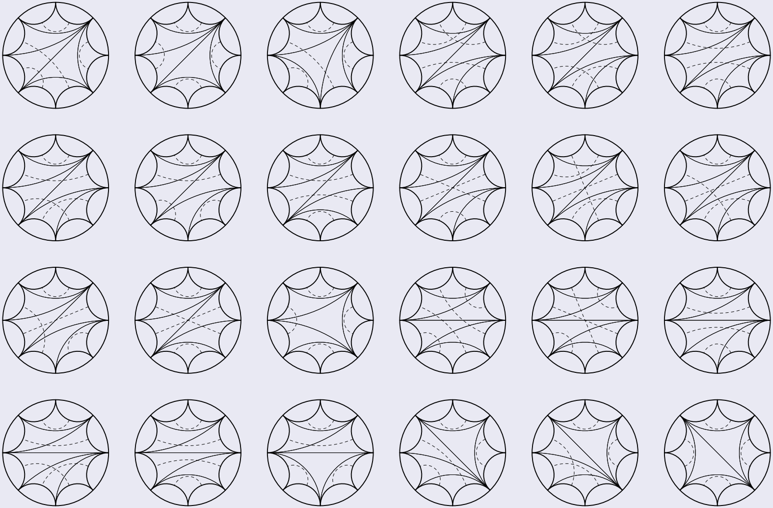
Samples of Results

The *eighty one* triangular maps with *six* faces (part two of four)



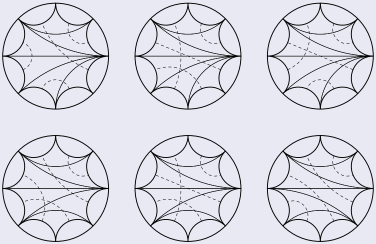
Samples of Results

The *eighty one* triangular maps with *six* faces (part three of four)



Samples of Results

The *eighty one* triangular maps with *six* faces (last part)



Samples of Results

The number of *pointed triangular maps* on an closed orientable surface (A062980).

$$\begin{aligned}\tilde{T}_3^{\bullet}(t) &= 5\,t^6 + 60\,t^{12} + 1105\,t^{18} + 27120\,t^{24} + 828250\,t^{30} \\ &\quad + 30220800\,t^{36} + 1282031525\,t^{42} + 61999046400\,t^{48} \\ &\quad + 3366961243750\,t^{54} + 202903221120000\,t^{60} + \dots\end{aligned}$$

Recurrence relation

If we note a_n the coefficient of t^{6n} the recurrence is as follows,

$$a_1 = 5 \quad \text{and} \quad a_n = 6n\,a_{n-1} + \sum_{k=1}^{n-2} a_k\,a_{n-k-1} \quad (n > 1)$$

The number of *unlabeled triangular maps* on an closed orientable surface. (**New !**)

$$\begin{aligned}\tilde{T}_3(t) &= 3\,t^6 + 11\,t^{12} + 81\,t^{18} + 1228\,t^{24} + 28174\,t^{30} + 843186\,t^{36} \\ &+ 30551755\,t^{42} + 1291861997\,t^{48} + 62352938720\,t^{54} \\ &+ 3381736322813\,t^{60} + 203604398647922\,t^{66} \\ &+ 13475238697911184\,t^{72} + 972429507963453210\,t^{78} \\ &+ 75993857157285258473\,t^{84} \\ &+ 6393779463050776636807\,t^{90} + \dots\end{aligned}$$

We have computed a generating series giving the number of combinatorial maps on a closed surface with a given number of edges and the list of degree for its faces (or by Poincaré duality, the list of degree for its vertices).

It comes thus with an infinite set of parameters t, u_1, u_2, u_3 and the coefficient of $t^{n_e} u_1^{n_1} u_2^{n_2} u_3^{n_3} \ldots$ is the number of combinatorial maps with n_e distinct edges and with n_1, n_2, n_3, etc, for the number of its *loops*, *spindles*, *triangles*, etc, or by Poincaré duality, n_1, n_2, n_3, etc for its number of vertices having degree 1, 2, 3, etc..

Samples of Results

The number of *unlabeled combinatorial maps* on an closed orientable surface with a given number n_e of edges (coeff. of t^{n_e}) and a given number n of k-gones (coeff. of u_k^n) . (**New !**)

$$M = (u_2 + {u_1}^2)\, t + ({u_1}^2 u_2 + u_1 u_3 + {u_2}^2 + 2\, u_4)\, t^2$$
$$+ \begin{pmatrix} u_3 {u_1}^3 + {u_1}^2 {u_2}^2 + {u_2}^3 + 3\, u_1 u_5 + 2\, {u_1}^2 u_4 \\ + 5\, u_6 + 2\, u_2 u_4 + 2\, u_3 u_1 u_2 + \underline{3\, {u_3}^2} \end{pmatrix} t^3$$
$$+ \begin{pmatrix} 9\, {u_1}^2 u_6 + {u_1}^2 {u_2}^3 + 9\, u_2 u_6 + 7\, u_5 u_3 + 2\, u_5 {u_1}^3 \\ + 9\, u_5 u_1 u_2 + 4\, {u_2}^2 u_4 + {u_1}^4 u_4 + 3\, {u_1}^2 {u_3}^2 + {u_2}^4 \\ + 7\, {u_4}^2 + 18\, u_8 + 5\, {u_1}^2 u_2 u_4 + 4\, {u_3}^2 u_2 + 15\, u_1 u_7 \\ + 8\, u_4 u_1 u_3 + u_3 {u_1}^3 u_2 + 3\, u_3 u_1 {u_2}^2 \end{pmatrix} t^4 + \dots$$