Some Recent Results on Solving and Factoring Differential and Difference Equations

(Fully Integrable Systems and Ore Modules)

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A brief introduction to

Key Laboratory of Mathematics Mechanization

Beginning

Wentsun Wu: *On the Decision Problem and the Mechanization of Theorem-Proving in Elementary Geometry*, Scientia Sinica (1978), 159-172.

Re-published in *Automated Theorem Proving, after 25 years*, AMS (1984)

Key steps

Theorem = hypotheses + conclusion

1. Translate the hypotheses to a system H of polynomial equations, and the conclusion to a polynomial g.
2. Decompose $\mathbf{V}(H)$ into irreducible components

$$\mathbf{V}(H) = V_1 \cup \cdots \cup V_r,$$

   where each V_i is represented by a set T_i of polynomials in triangular form.
3. Find the irreducible components corresponding to degenerating conditions.
4. Decide if g vanishes on the irreducible components that do not correspond to degenerating conditions.

Note: The algorithm used in step 2 is called the Ritt-Wu or the characteristic set method.

Mathematics Mechanization

From the preface of *Mathematics Mechanization* by Wu:

The subject tries to deal with mathematics in a constructive and algorithmic manner so that the reasonings become mechanical, automated, and as much as possible to be intelligence-lacking, with the results of lessening the painstaking heavy brain-labor.

Photos 1986, 2006

Projects related to differential and difference equations

1. Extending the characteristic set method to difference algebra
(*Gao and Yuan in collaboration with J. van der Hoeven*)
2. Finding algebraic solutions of first-order nonlinear ODE's with constant coefficients
(*Feng and Gao in collaboration with J. Cano*)
3. Decomposing linear partial differential and difference systems
(*Li and Wang in collaboration with M. Singer and M. Wu*)
4. Symbolic integration, decomposition of ODE's, rational solutions of OΔE, etc

Fully integrable systems and Ore modules

- Converting an integrable system to a fully integrable system
- Computing hyperexponential solutions of a fully integrable system
- Factoring reflexive Ore modules

Fully integrable systems

Linear ordinary difference equations

$F = \mathbb{C}(k)$, $\quad \sigma : k \mapsto k+1$ automorphism

Linear ordinary difference equation over F:

$$L(z) = \sigma^n(z) + a_{n-1}\sigma^{n-1}(z) + \cdots + a_1\sigma(z) + a_0 = 0, \quad a_i \in F$$

- $\dim_{\mathbb{C}} \mathrm{sol}(L) = n \iff a_0 \neq 0$
- when $a_0 = 0$

$$L(z) = \sigma(\overbrace{\sigma^{n-1}(z) + \sigma^{-1}(a_{n-1})\sigma^{n-2}(z) + \cdots + \sigma^{-1}(a_1)}^{\tilde{L}(z)}) = 0$$

$$\Downarrow$$

$$\tilde{L}(z) = 0$$

$$\Downarrow$$

reducing the order of equation

$$\Downarrow$$

$$\vdots$$

Matrix forms

$F = \mathbb{C}(k)$, $\quad \sigma : k \mapsto k+1$ automorphism,

$$\sigma(\mathbf{z}) = A\,\mathbf{z}, \qquad \text{with } \mathbf{z} = (z_1, \ldots, z_n)^T \text{ and } A \in F^{n\times n}.$$

- $\dim_{\mathbb{C}} \mathrm{sol}(\sigma(\mathbf{z}) = A\mathbf{z}) = n \iff A$ is invertible.
- when A is singular:

linear dep. among the row vectors of A

$\Downarrow \sigma(\mathbf{z}) = A\,\mathbf{z}$

linear relations in $\sigma(z_1), \ldots, \sigma(z_n)$

$\Downarrow \sigma^{-1}$

linear relations in $z_1, \ldots, z_n$, e.g. $(z_{d+1}, \ldots, z_n)^T = Q(z_1, \ldots, z_d)^T$,
substitute these relations into $\sigma(\mathbf{z}) = A\,\mathbf{z}$

$\Downarrow$

$$\sigma\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = B\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}, \quad R\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = 0$$

$\Downarrow$

$\vdots$

Example

$F = \mathbb{C}(k), \;\; \sigma : k \mapsto k+1,$

Consider the system

$$\sigma \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{k+1}{k} & 0 & 0 \\ \frac{k+1}{k} & 0 & 0 \\ \frac{k+1}{k} & -\frac{k}{k-1} & \frac{k}{k-1} \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$|A| = 0 \implies$

$$z_1 = z_2, \qquad \sigma \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{k+1}{k} & 0 \\ -\frac{1}{k(k-1)} & \frac{k}{k-1} \end{pmatrix}}_{B} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$$

$|B| \neq 0 \implies$ the solution space of $\sigma(\mathbf{z}) = A\,\mathbf{z}$ has dimension 2.

Partial case

$F = \mathbb{C}(k_1, k_2)$, $\sigma_1 : k_1 \mapsto k_1 + 1,\ k_2 \mapsto k_2$ and $\sigma_2 : k_1 \mapsto k_1,\ k_2 \mapsto k_2 + 1$.

A first-order system $\mathcal{A}$:

$$\sigma_1(\mathbf{z}) = A_1\mathbf{z},\ \ \sigma_2(\mathbf{z}) = A_2\mathbf{z} \quad \text{with } \mathbf{z} = (z_1 \ldots, z_n)^T,\ A_1,\ A_2 \in F^{n\times n},$$

is

- *integrable* if $\sigma_1(A_2)A_1 = \sigma_2(A_1)A_2$ (derived from $\sigma_1\sigma_2(\mathbf{z}) = \sigma_2\sigma_1(\mathbf{z})$)
- *fully integrable* if it is integrable and $A_1,\ A_2$ are invertible.

Note: If $\mathcal{A}$ is fully integrable, then its solution space has dimension n over $\mathbb{C}$.

From integrable to fully integrable

Idea: Let $\{\sigma_i(\mathbf{z}) = A_i\mathbf{z}\}_{i=1,2}$ be an integrable system where $A_i \in F^{n\times n}$

(1) If some A_j is singular, then

$$\text{solve } (y_1,\ldots,y_n)A_j = 0 \implies \text{ linear relations in } z_1,\ldots,z_n, \text{ e.g. } \begin{pmatrix} z_{d+1} \\ \vdots \\ z_n \end{pmatrix} = Q\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}.$$

(2) Reduce $\{\sigma_i(\mathbf{z}) = A_i\mathbf{z}\}_{i=1,2}$ to get

$$\sigma_i\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = B_i\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}, \quad R_i\begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = 0, \quad i = 1, 2.$$

$\implies \cdots$ Stop the reduction until both B_i are invertible and both $R_i = 0$.

Lemma: The difference system $\{\sigma_i(\mathbf{y}) = B_i\mathbf{y}\}_{i=1,2}$ is integrable if $R_1 = R_2 = 0$.

Example

Let $F = \mathbb{C}(n, k)$, $\sigma_n : n \mapsto n+1$, $\sigma_k : k \mapsto k+1$,

$$\{\sigma_n(\mathbf{z}) = A_n\mathbf{z},\ \sigma_k(\mathbf{z}) = A_k\mathbf{z}\}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-k-n-2}{k^2+k-1-2n-n^2} & 0 & \frac{-k^2+3n+2+n^2}{k^2+k-1-2n-n^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-k^2+n^2-2k+1+3n}{k^2+k-1-2n-n^2} & 0 & \frac{-k+n+1}{k^2+k-1-2n-n^2} \end{pmatrix},$$

$$A_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k^2+3k+2-n^2}{-n^2+k^2+k} & 0 & -\frac{2n}{-n^2+k^2+k} & 0 \\ 0 & \frac{k^2-n^2+3k+1-2n}{k^2+k-1-2n-n^2} & 0 & \frac{-2n-2}{k^2+k-1-2n-n^2} \end{pmatrix}$$

integrable, A_n singular $\implies$ not fully integrable.

Example (cont.)

Singularity of $A_n \implies$

$$z_1 = -\frac{-n^2+k^2+k}{k+n+1} z_2 - \frac{k^2-n-n^2}{k+n+1} z_3, \qquad z_4 = \frac{2k-n+1+k^2-n^2}{k+n+1} z_2 + \frac{k^2+k-1-2n-n^2}{k+n+1} z_3$$

Apply linear reduction to get

$$\sigma_n(z_2, z_3)^T = B_n(z_2, z_3)^T, \quad \sigma_k(z_2, z_3)^T = B_k(z_2, z_3)^T$$

where

$$B_n = \begin{pmatrix} \frac{-k-n-2}{k^2+k-1-2n-n^2} + \frac{(-k^2+3n+2+n^2)(2k-n+1+k^2-n^2)}{(k^2+k-1-2n-n^2)(k+n+1)} & \frac{-k^2+3n+2+n^2}{k+n+1} \\ \frac{2k-n+1+k^2-n^2}{k+n+1} & \frac{k^2+k-1-2n-n^2}{k+n+1} \end{pmatrix}$$

$$B_k = \begin{pmatrix} \frac{2k-n+1+k^2-n^2}{k+n+1} & \frac{k^2+k-1-2n-n^2}{k+n+1} \\ -\frac{k^2+3k+2-n^2}{k+n+1} & -\frac{(k^2+3k+2-n^2)(k^2-n-n^2)}{(-n^2+k^2+k)(k+n+1)} - \frac{2n}{-n^2+k^2+k} \end{pmatrix}$$

Both B_n and B_k are invertible. So the original system has a 2-dim solution space.

General setting

Let F be a field of characteristic zero.

Have

- $\delta_1, \ldots, \delta_\ell$ derivations on F
- $\sigma_{\ell+1}, \ldots, \sigma_m$ automorphisms of F.

Write $\Delta = \{\delta_1, \ldots, \delta_\ell,\ \sigma_{\ell+1}, \ldots, \sigma_m\}$

Assumption: All the maps in Δ commute pairwise.

Fully integrable systems

Consider a system $\mathcal{A}$:

$$\delta_i(\mathbf{z}) = A_i\mathbf{z},\ i = 1, \ldots, \ell \quad \text{and} \quad \sigma_j(\mathbf{z}) = A_j\mathbf{z},\ \ j = \ell + 1, \ldots m,$$

where $A_i, A_j \in F^{n\times n}$. The system $\mathcal{A}$ is integrable if

1. $\delta_i(A_j) = \delta_j(A_i) \quad (1 \le i < j \le \ell)$
2. $\sigma_i(A_j)A_i = \sigma_j(A_i)A_j \quad (\ell + 1 \le i < j \le m)$
3. $A_jA_i + \delta_i(A_j) = \sigma_j(A_i)A_j \quad (1 \le i \le \ell, \ \ \ell + 1 \le j \le m)$

The system $\mathcal{A}$ is fully integrable if it is integrable and $A_{\ell+1}, \ldots, A_m$ are invertible

Note: Can transform an integrable system to a fully integrable one in the same way.

Ore algebras

$\Delta = \{\delta_1, \ldots, \delta_\ell,\ \sigma_{\ell+1}, \ldots, \sigma_m\}$.

Let $\mathcal{O} = F[\partial_1, \ldots, \partial_\ell, \partial_{\ell+1}, \ldots, \partial_m]$ with commutation rules

1. $\partial_i \partial_j = \partial_j \partial_i$ for all $i, j \in \{1, \ldots, m\}$
2. $\partial_i f = f \partial_i + \delta_i(f)$ for all $f \in F$ and $i \in \{1, \ldots, \ell\}$
3. $\partial_j f = \sigma_j(f) \partial_j$ for all $f \in F$ and $j \in \{\ell + 1, \ldots, m\}$

From fully integrable systems to $\mathcal{O}$-modules

System: a fully integrable system $\mathcal{A}$:

$$\delta_i(\mathbf{z}) = A_i\mathbf{z},\ i = 1, \ldots, \ell \quad \text{and} \quad \sigma_j(\mathbf{z}) = A_j\mathbf{z} \quad j = \ell + 1, \ldots m,$$

where $A_i, A_j \in F^{n\times n}$.

Module: Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of F^n. Letting

$$\partial_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = -A_i^T \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \ i = 1, \ldots, \ell \quad \text{and} \quad \partial_j \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = \left(A_j^{-1}\right)^T \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \ j = \ell + 1, \ldots m.$$

Relation:

$(z_1, \ldots z_n)^T \in F^n$ is a solution of $\mathcal{A}$

$$\Updownarrow$$

$$\begin{cases} \partial_i(z_1\mathbf{e}_1 + \cdots + z_n\mathbf{e}_n) = 0 & i = 1, \ldots, \ell, \\ \partial_j(z_1\mathbf{e}_1 + \cdots + z_n\mathbf{e}_n) = z_1\mathbf{e}_1 + \cdots + z_n\mathbf{e}_n, & j = \ell + 1, \ldots, m \end{cases}$$

$\mathcal{O}$-modules associated with fully integrable systems

Let M be an $\mathcal{O}$-module.

- A submodule N of M is reflexive if

$$\partial_j \mathbf{v} \in N \implies \mathbf{v} \in N$$

  where $\ell + 1 \leq j \leq m$ and $\mathbf{v} \in M$.

- An $\mathcal{O}$-module is said to be reflexive if all its submodules are reflexive.

Proposition:

fully integrable systems	$\longleftrightarrow$	finite-dim reflexive $\mathcal{O}$-modules

Δ-extensions

Let F be a field and $\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_{\ell+1}, \ldots, \sigma_m\}$

Let E be a commutative F-algebra. Extend the maps in Δ from F to E:

derivations: $\delta_i : E \longrightarrow E \qquad i = 1, \ldots \ell,$

automorphisms: $\sigma_j : E \longrightarrow E \qquad j = \ell + 1, \ldots, m.$

(E, Δ) is called a Δ-extension of F if the extended maps commute pairwise.

Fundamental matrices

Definition: Let $\mathcal{A}$ be a fully integrable system of size n. An $n \times n$ matrix U over some Δ-extension of F is a fundamental matrix for $\mathcal{A}$ if

(i) U is invertible

(ii) Every column of U is a solution of $\mathcal{A}$.

Proposition: There exists a fundamental matrix for $\mathcal{A}$.

Correspondences

Assume

Fully integrable systems	$\mathcal{A}$	$\mathcal{B}$
Associated modules	$M_{\mathcal{A}}$	$M_{\mathcal{B}}$
Fundamental matrices	$U_{\mathcal{A}}$	$U_{\mathcal{B}}$

Definition: $\mathcal{A}$ and $\mathcal{B}$ are equivalent, denoted by $\mathcal{A} \sim \mathcal{B}$, if

$$\exists \text{ a square matrix } Q \text{ over } F, \quad U_{\mathcal{A}} = U_{\mathcal{B}} Q.$$

Propersition:

$M_{\mathcal{A}} \cong M_{\mathcal{B}}$	$\Leftrightarrow$	$\mathcal{A} \sim \mathcal{B}$
$0 \neq N \subsetneqq M_{\mathcal{A}}$	$\Leftrightarrow$	$\mathcal{A} \sim \mathcal{B}$ with each $B_i = \begin{pmatrix} \Box & \Box \\ 0 & \Box \end{pmatrix}$
$M_{\mathcal{A}} = N_1 \oplus N_2$	$\Leftrightarrow$	$\mathcal{A} \sim \mathcal{B}$ with each $B_i = \begin{pmatrix} \Box & 0 \\ 0 & \Box \end{pmatrix}$

where N, N_1, N_2 are submodules of $M_{\mathcal{A}}$.

Hyperexponential solutions of fully integrable systems

Hyperexponential elements

$\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_{\ell+1}, \ldots, \sigma_m\}$, (F, Δ) a Δ-field, (E, Δ) a Δ-ring extension of F

Definition: $0 \neq h \in E$ is *hyperexponential over F* if the $\delta_i(h)$ and $\sigma_j(h)$ are all linearly dependent on h over F.

Example:

	base field	rational	radical	transcendental
Partial differential	$\mathbb{C}(x, y)$	$\frac{x-y}{x^4 y}$	$\frac{2y+x^2}{\sqrt{xy}}$	$e^{x^3-y^2}$
Partial difference	$\mathbb{C}(m, n)$	$\frac{1}{m+n}$	$(\sqrt{-1})^m(-1)^n$	$\binom{m}{n}$
Differential-difference	$\mathbb{C}(x, n)$	$n + \frac{1}{x}$	$(-1)^n\sqrt{x}$	x^n

Definition: A nonzero vector $\mathbf{h} \in E^n$ is *hyperexp* if

$$\mathbf{h} = h\mathbf{v}, \qquad \text{where } h \in E \text{ is hyperexp. and } \mathbf{v} \in F^n$$

Problem and result

Problem: Given a fully integrable system, compute all its hyperexp sols.

Result: A method, which is recursive on the set $\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_{\ell+1}, \ldots, \sigma_m\}$.

Note: The method becomes an algorithm when

1. $F = \bar{\mathbb{Q}}(x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_m)$
2. $\delta_i = \frac{\partial}{\partial x_i}$ and $\sigma_j : x_j \mapsto x_j + 1$.

Structure of hyperexp sols

Proposition:

All hyperexp sols of a fully integrable system can be partitioned into

$$\{h_1 V_1 \mathbf{d}_1\} \cup \cdots \cup \{h_s V_s \mathbf{d}_s\},$$

where

- h_i is hyperexp;
- V_i is a matrix over F;
- $\mathbf{d}_i$ is an arbitrary nonzero vector over C_F.

Extensible hyperexponential elements

Let $\Delta_1, \Delta_2 \subset \Delta$ be nonempty and disjoint. Let E_1 be a Δ_1-extension of F.
Let $h_1 \in E_1$ be hyperexponential w.r.t. Δ_1.

Definition: We say that h_1 is extensible with respect to Δ_2 if there exist

- a $(\Delta_1 \cup \Delta_2)$-extension E_2 of F, and
- an element $h_2 \in E_2$, which is hyperexponential w.r.t. $(\Delta_1 \cup \Delta_2)$

such that

$$\frac{\theta_1(h_2)}{h_2} = \frac{\theta_1(h_1)}{h_1} \quad \text{for all } \theta_1 \in \Delta_1.$$

Example

Let $F = \mathbb{C}(x, y)$ and $\Delta = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$.

Then

$$h = \exp\left(f(x, y)\, dx\right) \quad \text{where } f \in F,$$

is extensible w.r.t $\frac{\partial}{\partial y}$ iff

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial y}$$

has a solution in F.

If $f = \frac{y}{x}$, then $\frac{\partial z}{\partial x} = \frac{1}{x}$ has no solution in F, and so

$$h = \exp\left(\int \frac{y}{x}\, dx\right) = x^y$$

is not extensible w.r.t. $\frac{\partial}{\partial y}$.

Computing hyperexp sols of $\{\theta(\mathbf{z}){=}A_\theta \mathbf{z}\}_{\theta\in\Delta}$

1. If $|\Delta| = 1$: known methods. Otherwise partition $\Delta = \Delta_1 \cup \Delta_2$.
2. Find hyperexp sols of $\{\theta_1(\mathbf{z}){=}A_{\theta_1}\mathbf{z}\}_{\theta_1\in\Delta_1}$

$$\begin{array}{cccc} (\{h_1 V_1 \mathbf{d}_1\} \cup \ \ldots, & \{h_s V_s \mathbf{d}_s\} \cup \{h_{s+1}V_{s+1}\mathbf{d}_{s+1}\} \cup \cdots \cup \{h_t V_t \mathbf{d}_\mathbf{t}\} \\ \downarrow \text{ext.} & \downarrow \text{ext.} \\ \bar{h}_1, \quad \ldots, & \bar{h}_s \end{array} \qquad (1)$$

3. Determine extensibility w.r.t. Δ_2.
4. Substitute $\mathbf{z} = \bar{h}_i V_i \mathbf{d}_i$, $i = 1, \ldots, s$, into $\{\theta_2(\mathbf{z}){=}A_{\theta_2}\mathbf{z}\}_{\theta_2\in\Delta_2}$.

   [**Note:** The substitution neither misses any hyperexp. sols nor introduces any coefficients outside F due to the replacement of h_i by $\bar{h}_i$]
5. Transform new systems in $\mathbf{d}_i$ to fully integ. systems $\{\theta_2(\mathbf{y}){=}B^{(i)}_{\theta_2}\mathbf{y}\}_{\theta_2\in\Delta_2}$ over $\mathrm{const}_{\Delta_1}(F)$, where $i = 1, \ldots, s$.
6. Compute hyperexp sols of $\{\theta_2(\mathbf{y}){=}B^{(i)}_{\theta_2}\mathbf{y}\}_{\theta_2\in\Delta_2}$ and combine with (1).

Example

$F=\mathbb{Q}(e)(x,y)$, $\sigma(x)=x+1$, $\delta=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$. Compute hyperexp sols of

$$\sigma(\mathbf{z})=\begin{pmatrix} 0 & \frac{1}{y} & -x\,e & e+1 \\ -y\,e & e+1 & 0 & y\,e \\ 0 & 0 & 0 & \frac{1}{x+1} \\ 0 & 0 & -x\,e & e+1 \end{pmatrix}\mathbf{z},\quad \delta(\mathbf{z})=\begin{pmatrix} -\frac{1}{y} & -\frac{4}{2y-1} & \frac{x}{y} & \frac{2y-1+4y^2}{y(2y-1)} \\ 0 & 0 & 0 & 1 \\ -\frac{4y}{x(2y-1)} & 0 & \frac{4yx-2y+1}{x(2y-1)} & \frac{4y}{x(2y-1)} \\ 0 & -\frac{4}{2y-1} & 0 & \frac{4y}{2y-1} \end{pmatrix}\mathbf{z}.$$

$$\mathbf{z}_1=1\cdot\begin{pmatrix} \frac{1}{y} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \xrightarrow{\text{subs}} \quad \delta\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -\frac{4}{2y-1} & \frac{4y}{2y-1} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$\mathbf{z}_2=e^x\cdot\begin{pmatrix} \frac{1}{y\,e} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x\,e} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \xrightarrow{\text{subs}} \quad \delta\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ -\frac{4}{2y-1} & \frac{2y+1}{2y-1} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Note: c_1 and c_2 are free of x.

Solutions:

$$\begin{pmatrix} 2 \\ y \\ \frac{1}{x} \\ 1 \end{pmatrix},\ e^{2y}\begin{pmatrix} \frac{1}{y}+2 \\ 1 \\ \frac{2}{x} \\ 2 \end{pmatrix},\ e^{x+y}\begin{pmatrix} \frac{1}{y}+2e \\ e \\ \frac{2}{x} \\ 2e \end{pmatrix},\ e^{x-y}\begin{pmatrix} 1+e \\ ye \\ \frac{1}{x} \\ e \end{pmatrix}.$$

Determining submodules of finite-dimensional reflexive Ore modules

Problem and result

Let M be a reflexive and finite-dimensional $\mathcal{O}$-module with an F-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and.

$$\partial_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_m \end{pmatrix} = B_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_m \end{pmatrix} \quad B_i \in F^{n\times n}, i = 1, \ldots, m$$

Want: Determine all the submodules of M.

Result: A generalization of Beke's method for determining the submodules of M.

Generalization

Let M be a reflexive $\mathcal{O}$-module of finite dimension n over F.

Proposition A: Let $\mathbf{v}_1, \cdots, \mathbf{v}_d \in M$ be F-linearly independent. Then

$$F\mathbf{v}_1 + \cdots + F\mathbf{v}_d \text{ is a submodule} \Leftrightarrow F(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_d) \text{ is a submodule of } \wedge^d M.$$

Proposition B: Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an F-basis of M and $\mathcal{A}$ be the associated system. Let

$$\mathbf{w} = f_1\mathbf{e}_1 + \cdots + f_n\mathbf{e}_n$$

with $f_i \in F$. Then

$$F\mathbf{w} \text{ is a submodule} \Leftrightarrow \exists h \text{ hyperexponential} \quad h(f_1, \ldots, f_n)^T \text{ solves } \mathcal{A}.$$

Determine all d-dimensional submodules of M

Form $\wedge^d M$ and an associated system $\mathcal{A}$

$\Downarrow$

hyperexponential solutions of $\mathcal{A}$

$\Downarrow$ Prop B

1-dim submodules of $\wedge^d M$

$\Downarrow$ Linear algebra

1-dim submodules of the form $F(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_d)$

$\Downarrow$ Prop A

d-dim submodules of M

Example

Let $F = \mathbb{C}(x, k)$, $\Delta = \{\frac{d}{dx},\ \sigma_k : k \mapsto k + 1\}$, and $\mathcal{O} = F[\partial_x, \partial_k]$.

M is an $\mathcal{O}$-module with an F-basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

$$\partial_x \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-x^3-x^2k+2x^2+xk+k^2x+k^2+k^3}{x^2(-x+k)} & \frac{2(x^2-x-k^2)}{(x-k)x} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-x^3-x^2k+x^2+3xk+2x+k^2x+4k^2+5k+2+k^3}{x^2(-x+k+1)} & -\frac{2(-x^2+x+k^2+2k+1)}{(-x+k+1)x} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}$$

and

$$\partial_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(x-k)x^2}{x-k-2} & 0 & \frac{2x(x-k-1)}{x-k-2} & 0 \\ -\frac{2x(x^2-2xk-3x+k^2+2k)}{(x-k-2)^2} & -\frac{(x-k)x^2}{x-k-2} & \frac{2(x^2-2xk-4x+k^2+3k+2)}{(x-k-2)^2} & \frac{2x(x-k-1)}{x-k-2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}.$$

Find all two-dimensional submodules of M.

$$N_i = \{a_1\mathbf{u}_{i,1} + a_2\mathbf{u}_{i,2} \mid a_1, a_2 \in F\}, \quad i = 1, 2, 3, 4.$$

For N_1,

$$\begin{aligned}
\mathbf{u}_{1,1} \;=\; & \Big((2c_1x^2k + c_1k^2x + c_4x^5 + c_1x^2 + 2c_1x^4 + c_2x^2k - 2x^3c_2k - 3x^3c_1k - 2x^4c_4k \\
& +x^2c_2k^2 + xk^3c_1 + x^4c_2 + c_4x^3k^2 - c_2x^3 - 2c_4x^4 - 3c_1x^3 + c_4x^3 + 2c_4x^3k)/ \\
& (x(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k \\
& -c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))\Big)\mathbf{e}_1 \\
& -\Big((c_4x^3 + c_1x^2 - kc_4x^2 - c_4x^2 - c_1xk - c_1x)(x - k)/ \\
& (2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 \\
& +2k^2c_4 - 2c_1k - c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2)\Big)\mathbf{e}_2 + \mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_{1,2} \;=\; & \Big((c_1x^4 - 2c_1x^2 + c_4x^5 + c_1x - 2c_4x^3 - c_1x^3 - c_4x^4 + c_2x^3 - 2c_2x^2 \\
& +2c_4x^2 + c_1k^2 + 2c_1k^3 + c_1k^4 - 2c_2x^2k + 3c_1k^2x - 4c_1x^2k + 5kc_4x^2 + k^2c_4x + c_2xk \\
& +3c_1xk - 4c_4x^3k - 2c_4x^3k^2 - 2c_1x^2k^2 + 2c_4xk^3 + 3c_4x^2k^2 + c_4xk^4 + c_2xk^2)/ \\
& (x(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 \\
& +2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))\Big)\mathbf{e}_1 \\
& -\Big((c_4x^3 - c_2x^2 + kc_4x^2 + c_2xk - 2k^2c_4x + c_2x - 2xc_4 + c_1xk - 4kc_4x - c_1 - c_1k^2 - 2c_1k)(x - k)/ \\
& ((2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 \\
& +2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))\Big)\mathbf{e}_2 + \mathbf{e}_4,
\end{aligned}$$

Summary

(i) **From integrable systems to fully integrable ones**

- *understand the algorithm from a module-theoretic viewpoint*

(ii) **Hyperexponential solutions of fully integrable systems**

- *develop efficient algorithms for ordinary cases*

(iii) **Factoring fully integrable systems (module-theoretic version)**

- *organize submodules with respect to isomorphisms*