Polynomial approximation and floating-point numbers

Algorithms Project Seminar

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Arenaire team

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Contents

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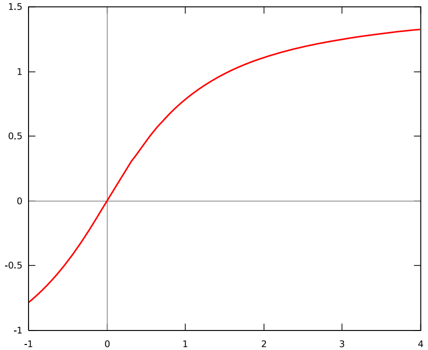
Why an approximation ?



Graph of $f : x \mapsto \arctan(x)$

- Let f be a real valued function : $f : \mathbb{R} \to \mathbb{R}$.

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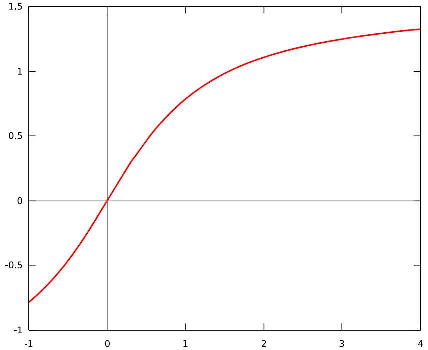

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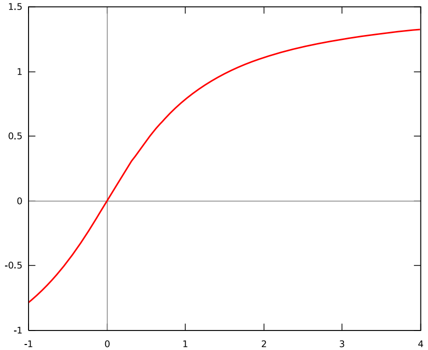

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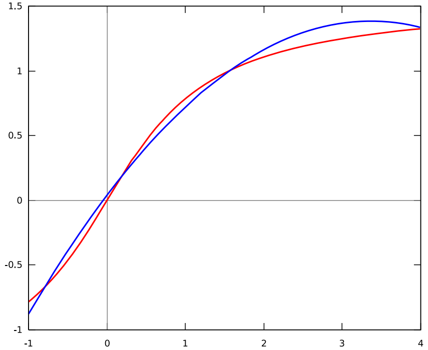
- We can only compute approximated values and hopefully bound the approximation error.

About the error of approximation



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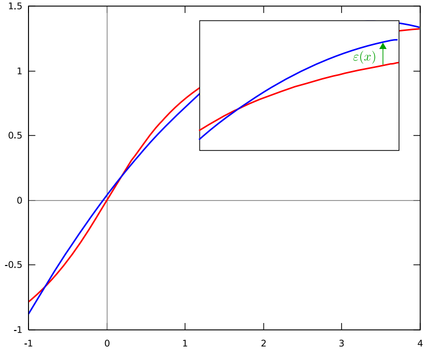

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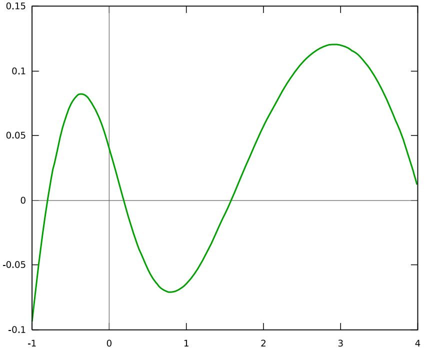

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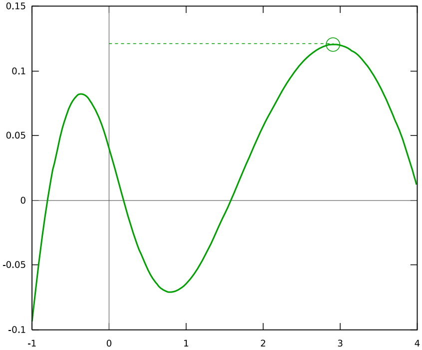

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$$\|\varepsilon\|_\infty = \max_{x \in [a, b]} \{|\varepsilon(x)|\}$$

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- Natural question : what degree should have a polynomial to give a suitable approximation ?

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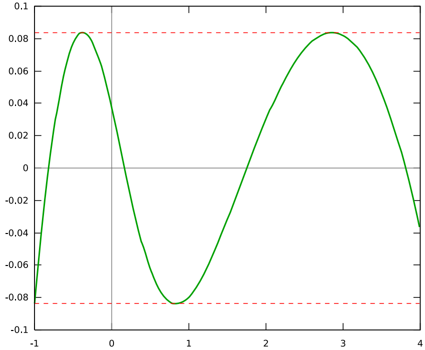

$n+2$ oscillations

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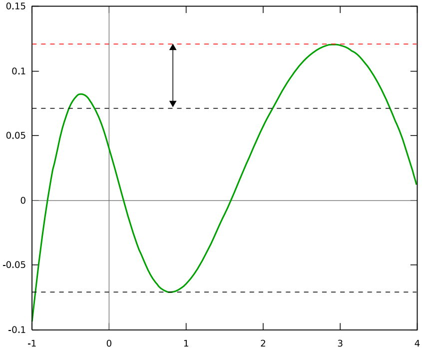

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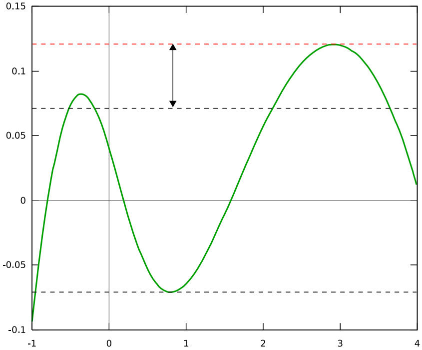

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- Remez' algorithm : given n, computes the optimal polynomial of degree $\leq n$ (called minimax).

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- IEEE `double` format : $\beta = 2$, $t = 53$, and $e \in [\![-1074, 971]\!]$.
- From now on, we will assume that $[e_{\min}, e_{\max}] = [-\infty, +\infty]$.

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- $\widehat{p}$ may be far from being optimal.
- Example with $f(x) = \log_2(1 + 2^{-x})$, $n = 6$, on $[0; 1]$ with single precision coefficients (24 bits).

Minimax	Naive method	Optimal
$8.3 \cdot 10^{-10}$	$119 \cdot 10^{-10}$	$10.06 \cdot 10^{-10}$

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- N. Brisebarre, J.-M. Muller and A. Tisserand have proposed an approach by linear programming (the implementation relies on P. Feautrier's tool PIP).

Method of Brisebarre, Muller and Tisserand

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- But :
  - its time is exponential ;
  - it is very sensitive to some parameters.
- We developed a new method :
  - fast (it is proven to run in polynomial time) ;
  - heuristic (there is no proof that the result is always tight) ;
  - with good practical results.

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- Problem : given n and a floating-point format, find (one of) the polynomial(s) p of degree $\leq n$ with floating-point coefficients minimizing $\|p - f\|_\infty$.

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- Remark : the existence is still ensured. The unicity may be lost.
- A simplification : we may try to guess the value of each e_i (assuming that the coefficients of p and p^* have the same order of magnitude)
  $\hookrightarrow$ if e_i is correctly guessed, we are reduced to find $m_i \in \mathbb{Z}$ such that

$$\left\| f(x) - \sum_{i=0}^{n} m_i \cdot \beta^{e_i} x^i \right\|_\infty$$

  is minimal.

Description of our method

Our goal : find p approximating f and with the following form :

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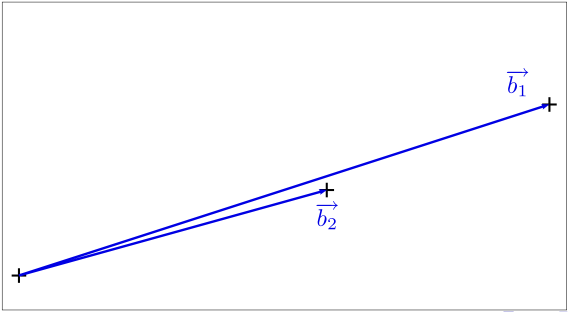
  - Rewritten with vectors :

$$\underbrace{m_0 \begin{pmatrix} \beta^{e_0} \\ \beta^{e_0} \\ \vdots \\ \beta^{e_0} \end{pmatrix} + \cdots + m_n \begin{pmatrix} \beta^{e_n} \cdot x_0^n \\ \beta^{e_n} \cdot x_1^n \\ \vdots \\ \beta^{e_n} \cdot x_n^n \end{pmatrix}}_{\Gamma \text{ of the form } \mathbb{Z}\overrightarrow{b_0}+\mathbb{Z}\overrightarrow{b_1}+\cdots+\mathbb{Z}\overrightarrow{b_n}} \simeq \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\overrightarrow{t} \in \mathbb{R}^{n+1}} \quad .$$

Notions about lattices

Let $(\overrightarrow{b_1}, \cdots, \overrightarrow{b_n})$ be a basis of a real vector space.



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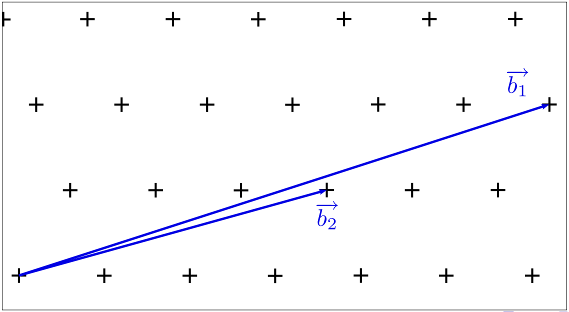

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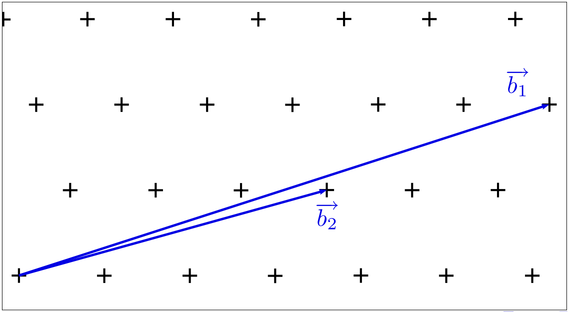

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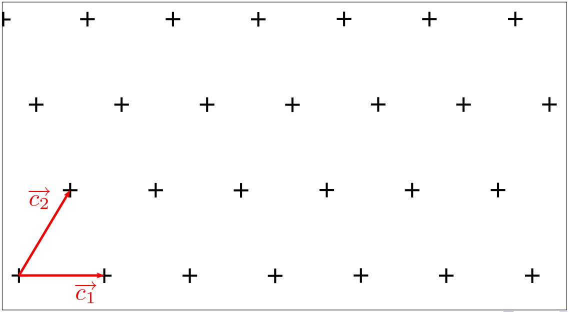

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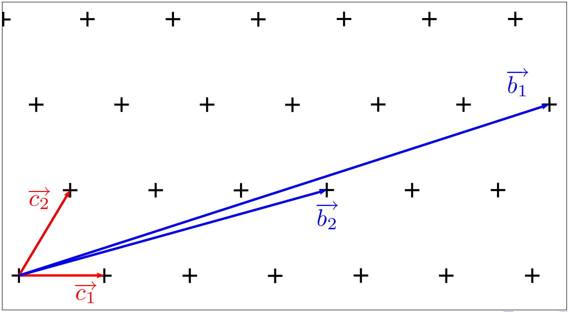

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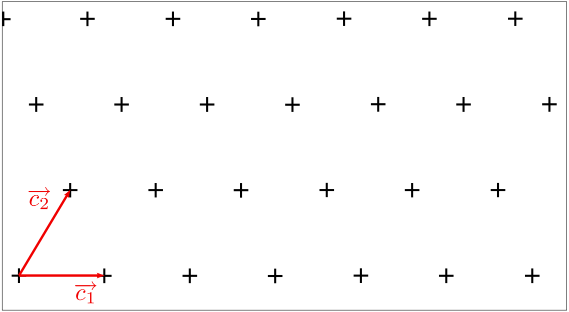

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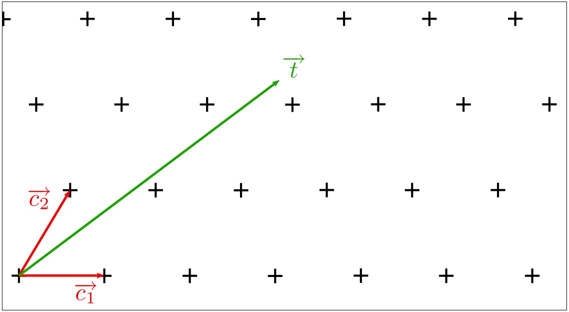

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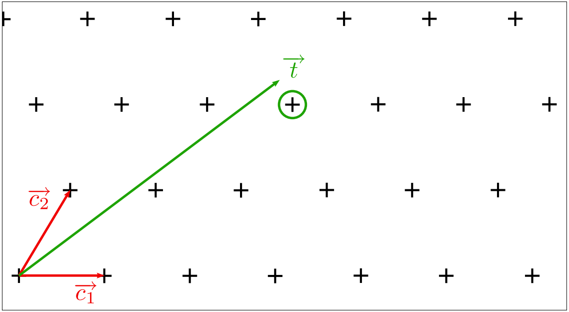

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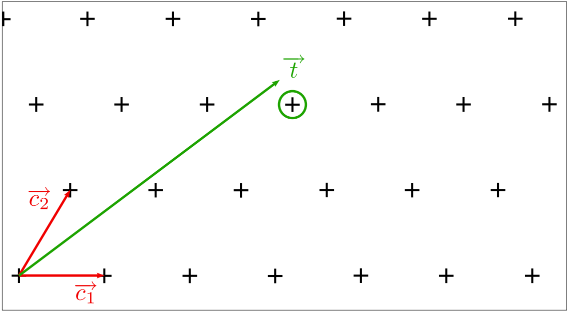

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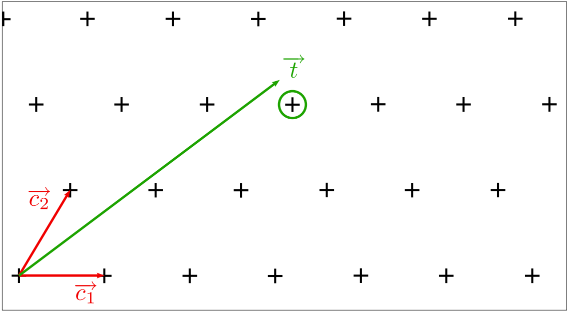

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- LLL terminates in at most $O(n^6 \ln^3 B)$ operations with $B = \max ||b_i||^2$.
- Very good practical results compared to the theoretical bounds.

LLL reduction

- Gram-Schmidt orthogonalization : to any basis $(b_1, \cdots, b_n)$ of a vector space is associated an orthogonal basis $(b_1^*, \cdots, b_n^*)$ such that $\mathrm{Span}(b_1, \cdots, b_j) = \mathrm{Span}(b_1^*, \cdots, b_j^*)$ for all j.

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- Idea of LLL algorithm : control the Gram-Schmidt basis to make $b_1^* = b_1$ minimal among the vectors of the orthogonal basis.
- Babai's algorithm uses the LLL algorithm to solve an approximation of CVP.

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  - other coefficients are `double extended` numbers.
- A `double extended` number has 64 bits of mantissa.
- He actually wants to have approximately 74 correct bits. (i.e. $\varepsilon \simeq 5.30\mathrm{e}{-23}$)

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5.30e−23	40.1e−23	0.07897e−23

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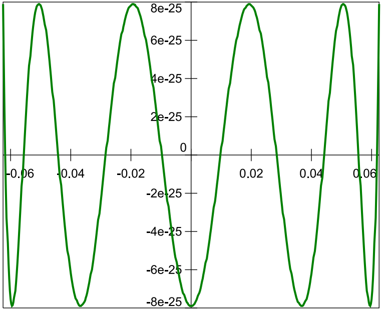

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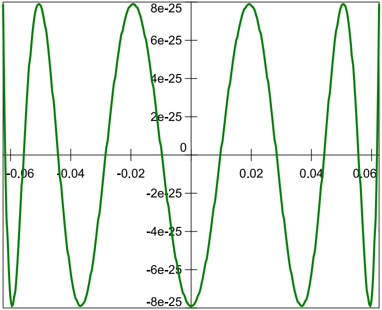

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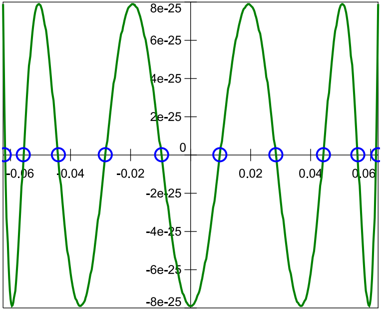

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- Chebyshev's theorem gives $n + 1$ such points.

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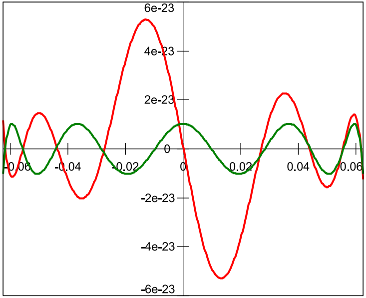

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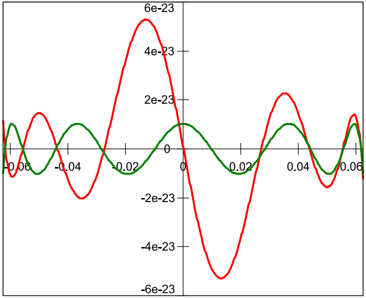

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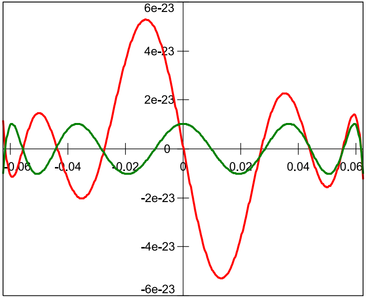

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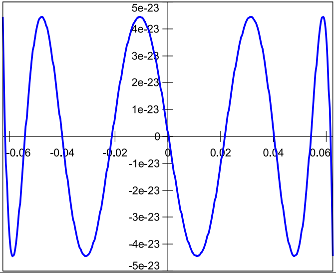

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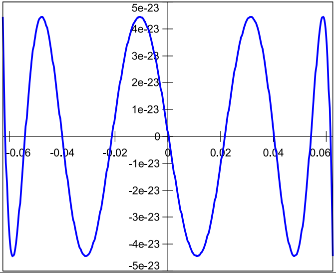

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- This time, our polynomial p_2 gives an error of 4.44e−23 and is practically optimal.

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- The algorithm is flexible : each coefficient may use a different floating-point format, one may search polynomial with additional constraints.

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