Behaviour of the Newton process in presence of a multiple isolated root, consequences and applications

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1 – Position of the problem

Let $f = (f_1, \ldots, f_n) = 0$ be a system of

- polynomial functions
- analytic functions defined on a connected open subset $\mathcal{U} \subset \mathbb{C}^n$

in n complex variables ;

Let ζ a zero of this system of finite multiplicity, and thus isolated in $f^{-1}(\{0\})$.

Goal : approximate numerically ζ with the classical Newton's operator

$$
\begin{aligned}
N_f : \mathbb{C}_s^n &\longrightarrow \mathbb{C}_s^n \\
z &\longmapsto z - Df(z)^{-1} f(z)
\end{aligned}
$$

If ζ is a regular root of the system, let us mention Smale's γ-theorem :

Theorem 1 (γ-Theorem) *Let*

$$\psi(u) = 1 - 4u + u^2$$

$$\gamma(f, \zeta) := \sup_{k \geqslant 2} \left(\frac{\| Df(\zeta)^{-1} D^k f(\zeta) \|}{k!} \right)^{\frac{1}{k-1}}$$

if a given $z_0 \in \mathbb{C}^n$ *satisfies*

$$u := \gamma(f,\zeta)\|z_0 - \zeta\| < \frac{5-\sqrt{17}}{4}$$

then the Newton sequence, initialized at z_0*, is well-defined and converge quadratically to* ζ *with*

$$\|z_k - \zeta\| \le \left(\frac{u}{\psi(u)}\right)^{2^k-1} \|z_0 - \zeta\|, \; k \ge 0$$

Reference :

L. Blum, F. Cucker, M. Shub, and S. Smale, *Complexity and real computation*, Springer-Verlag, 1998

However, in the singular case, we can observe experimentally that, if Newton's algorithm converge to ζ, then the convergence is **linear** due to a **geometric grow in one direction of space**.

What we propose here :

- Geometric caracterisation of directions of linear convergence
- Quantitative analysis of the behaviour of Newton's method with a γ-theorem in the spirit of the preceeding result.

2 – Schröder and Rall's contribution

2.1 – Schröder's operator

f a complex polynomial (or holomorphic function)
ζ a zero of f of multiplicity $\mu < +\infty$,that is :

$$f(\zeta) = f'(\zeta) = \ldots = f^{(\mu-1)}(\zeta) = 0 \quad \text{and} \quad f^{(\mu)}(\zeta) \neq 0$$

suppose the Newton's iterates $(z_k)_{k\geqslant 0}$ converge to ζ
Rate of convergence : $\lim_{k\to+\infty} \eta_k$ where $\eta_k := \varepsilon_{k+1} - \varepsilon_k$ and $\varepsilon_k := z_k - \zeta$

$$\varepsilon_{k+1} = \left(\frac{\mu-1}{\mu}\right)\varepsilon_k + \mathcal{O}(\varepsilon_k^2)$$

the convergence of the z_k's is geometric with a rate $\frac{\mu-1}{\mu}$

Schröder : If the Corrected Newton's method defined by $N_{\mu,f}(z) := z - \mu\frac{f(z)}{f^{(\mu)}(z)}$ converge, then the convergence is quadratic.

2.2 – Multivariable case : Rall's flag

$f = (f_1, \ldots, f_n) = 0$ a system of n polynomial (or analytic) functions of n complex variables $z_1, \ldots, z_n$;
$\zeta = (\zeta_1, \ldots, \zeta_n)$ a zero of this system of multiplicity $1 < \mu < +\infty$;
μ is the dimension of the **local algebra** $\mathbb{C}[x_{1:n}]_\zeta/(f_{1:n})$ in the polynomial case and $\mathbb{C}\{x_{1:n}\}_\zeta/(f_{1:n})$ in the analytic case.
Rall defined the flag of vector spaces at the root :

$$N_1 = \ker Df(\zeta) \supset N_2 := N_1 \cap \ker D^2 f(\zeta) \supset \ldots \supset N_\mu = \{0\}$$

where the $D^k f(\zeta),\ 1 \leqslant k \leqslant \mu$ are view has linear operators. Thus, the kernel of $D^2 f(\zeta)$ is the vector space

$$\{X \in_\zeta \mathbb{C}^n;\ D(Df)(\zeta)(X, .) = 0\}$$

He got a unique decomposition of the source space :

$$\begin{aligned}\mathbb{C}^n &= N_1^\perp \oplus N_1 \\ &= N_1^\perp \oplus (N_2^\perp \oplus N_2) \\ &= N_1^\perp \oplus \ldots \oplus N_{\mu-1}^\perp \oplus N_{\mu-1}\end{aligned}$$

If we denote by p_k and $p_k^\perp$ the orthogonal projections onto N_k and $N_k^\perp$ respectively, then Rall's conjecture can be expressed has follow :

$$\|p_k^{\perp}(\varepsilon_1 - \frac{k-1}{k}\varepsilon_0)\| = \mathcal{O}(\|\varepsilon_0\|^2),\ 1 \leqslant k \leqslant \mu$$

where $\varepsilon_0 = z_0 - \zeta$ and $\varepsilon_1 = N_f(z_0) - \zeta$.
Thus, if Rall's conjecture was correct, we could define the sequence $(y_k)_{k\geqslant 1}$:

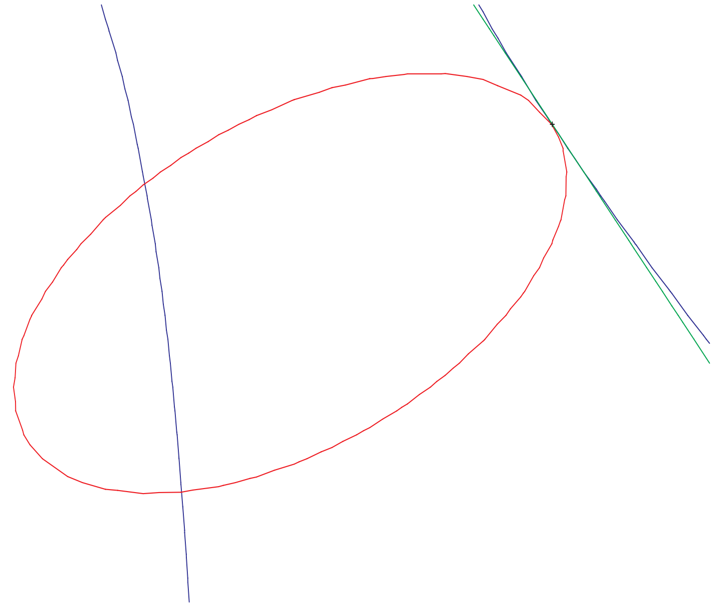
$$y_k = (p_1^{\perp}(z_k), p_2^{\perp}(2z_k - z_{k-1}), \ldots, p_{\mu-1}(\mu z_k - (\mu - 1)z_{k-1}))$$

and state $\|y_k - \zeta\| = \mathcal{O}(\|z_{k-1} - \zeta\|^2)$.
Unfortunately, this construction works only for the case **simple-double zeroes** and the proof he gave is wrong in general.
References :

E. Schröder, *Über unendlich viele algorithmen zur auflôsung der gleichungen*, Math. Annalen $2, 317 - 365$ (1870)

L. B. Rall, *Convergence of the Newton process to multiple solutions*, Numerische Mathematik $9, 23 - 27$ (1966)

all's example :



$$f_1 = x_1^2 - x_1x_2 + x_2^2 + x_1 - 2$$

$$f_2 = 3x_1^2 + 2x_1x_2 + 2x_2 - 7$$

= $(1, 1)$ is a root of multiplicity 2

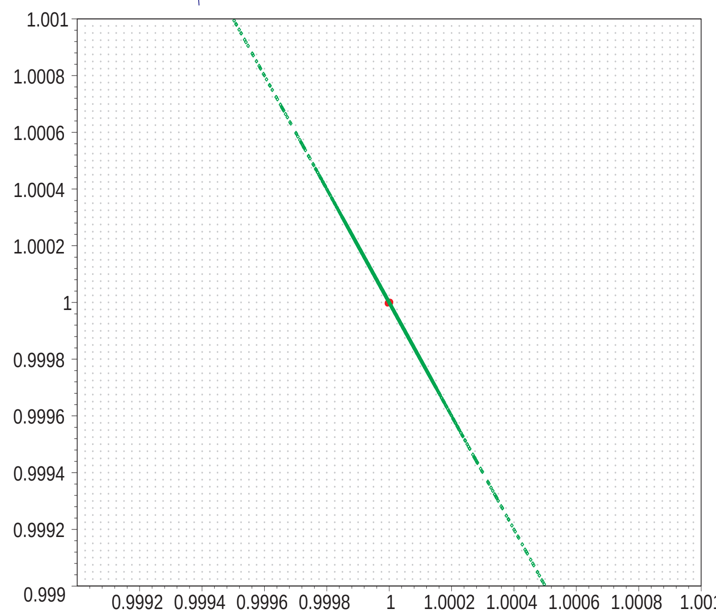
$$Df(1,1) = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}$$

$$D^2f(1,1) = \begin{pmatrix} 2 & -1 & -1 & 2 \\ 6 & 2 & 2 & 0 \end{pmatrix}$$

$$\ker Df(1,1) = \{2x_1 + x_2 = 0\}$$

$$Rad = \{(0,0)\}$$

re $\|p_1(2\varepsilon_1 - \varepsilon_0)\| = \mathcal{O}(\|\varepsilon_0\|^2)$, the Newton ite-
s converge quadratically to the tangent line $(1,1) +$
$Df(1,1)$ and the rate of convergence over this line is
.



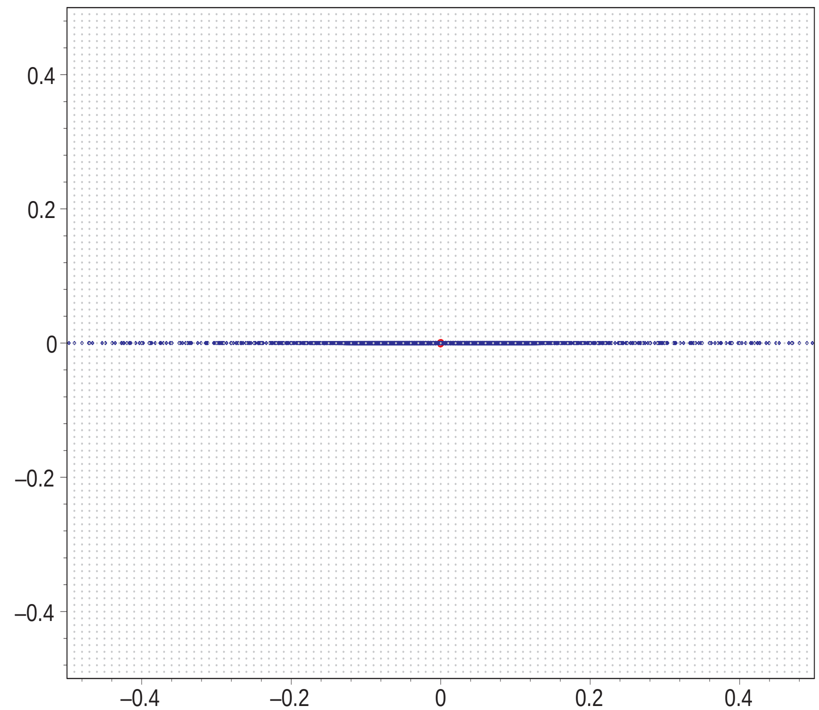
ounter-example to Rall's conjecture : Whitney's pleat

$$f_1 = x_1^3 + x_1 x_2, \quad f_2 = x_2$$

$$\Sigma(f) = \Sigma^1(f) = \{3x_1^2 + x_2 = 0\}$$

$$T_{(0,0)}\Sigma^1(f) = \{x_2 = 0\} = \ker Df(0,0)$$

e singular locus of f is the set of points of corank 1. e can show that the rate of convergence given by 's result is not the right one : the points in blue correspond to the rate $1/2$ while the red ones correspond /3.



3 – Corank 1 zeroes

families of singularities can be distinguished :

- Simple-double points
- Whitney's gather and generalized Whitney's singularities (also called Morin's singularities)

3.1 – The simple-double zeros case

finition 1 *ζ is called a* **simple-double zero** *of f iff*

- $\ker Df(\zeta)$ *is 1-dimensional over the ground field, spanned by a unitary vector v ;*
- $D^2 f(\zeta)(v,v) \notin \mathrm{im} Df(\zeta)$

ample 1 ➡ *Rall's example belongs to this class :*

$$D^2 f(1,1).[(u,-2u),(u,-2u)] = -2(5u^2,u^2) \notin \mathrm{im} Df(1,1) = \{x_1 - 4x_2 = 0\}$$

The Whitney's fold $(x_1,x_2) \to (x_1^2,x_2)$: the projection of the $\mathbb{R}^3$'s sphere onto the real plane.

The only quantitative result for this type of zeroes is due to Dedieu and Shub (1998), it generalize Smale's γ-theory which applies uniquely to regular zeros.

J. P. Dedieu, M. Shub, *On simple double zeros and badly conditioned zeros of analytic functions of n variables*, Math. Comp., pages 319-327, 2001.

3.2 – The (generalized) Whitney's singularities case

The Whitney's gather has already been treated ;

Morin's singularities : defined by the generalized Whitney's map

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1}, x_1 x_n + x_2 x_n^2 + \ldots + x_{n-1} x_n^{n-1} + x_n^{n+1})$$

$$Df(0, \ldots, 0) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Sigma^1(f) = \{x_1 + 2x_2 x_n + \ldots + (n-1) x_{n-1} x_n^{n-2} + (n+1) x_n^n = 0\}$$

$$T_{(0,\ldots,0)} \Sigma^1(f) = \{x_1 = 0\} \supseteq \ker Df(0, \ldots, 0)$$

In such a situation, *G. Lecerf's deflation algorithm* is powerful.

G. Lecerf, *Quadratic Newton iteration for systems with multiplicity*, Found. Comput. Math., $2(3) : 247 - 293, 2002$

M. Giusti, G. Lecerf, B. Salvy, and J. C. Yakoubsohn, *On location and approximation of clusters of zeroes : case of embedding dimension one*, (2004)

3.3 – Principal results

α-theory	Dedieu and Shub (1998)	quantitative results in the vein of Smale's α-theory, for simple-double zeros
eflation	Ojika, Watanabe, Mitsui (1983) ; Ojika (1997) ; Lecerf (2002) ; Verschelde (2004)	deflation consist in differentiating well chosen equations, both numeric and symbolic
orrected Newton methods	Reddien (1978, 1979) ; Decker and Kelley (1980) ; Griewank (1980, 1983, 1985)	rate $1/2$ for simple-double zeroes ; extension to Banach spaces ; precise the convergence domain
ordering techniques	Griewank (1985) ; Kunkel (1988, 1989) ; Govaerts (1997)	a system with a double zero is transformed into one woth a simple solution, it deals with high multiplicities
egularization techniques	Allgower, Bömer, Hoy, Janovsky (1999)	regularization of the Newton's correction, for corank m but first order singularities
gebraic topology	Kravanja, Van Barel (2000) + Sakurai (2003) ; Stenger (1975)	numerical integration and residue formula ; root counting based on topological degree theory
obal techniques	Faugere (1999) ; Lecerf (2002) ; Sommese, Verschelde (1996, 2000, 2002)	commutative algebra, Gröbner basis computation ; geometric solving ; homotopy continuation

4 – Corank at least 1 singularities of generic maps

4.1 – First order singularities

The singular locus $\Sigma(f) := \{z \in \mathbb{C}^n | det(Df(z)) = 0\}$ has a natural subsets partition

$$\Sigma^i(f) := \{z \in \mathbb{C}^n | dim_{\mathbb{C}} \ker Df(z) = i\}$$

In the case of Whitney's pleat, the stratum of corank 1 points is a parabola, thus a smooth subvariety ; in general, it won't be the case.

4.2 – Thom-Boardman's varieties

For the Whitney's gather : since $T_{(0,0)}\Sigma^1(f) = \ker Df(0,0)$, the origin is an **over-exceptional critical point** (in the sense of Thom) ; it will be denote by : $0 \in \Sigma^1(f_{|\Sigma^1(f)}) =: \Sigma^{1,1}(f)$.
In **Thom-Boardman stratification**, at each level, the stratum containing the singular point is locally a subvariety.
This introduce the notion of **generic** or **transversal** map.

References

R. Thom, *Les singularités des applications différentiables*, Annales de l'institut Fourier $6, 43-87, 1956$

J. M. Boardman, *Singularities of differentiable maps*, Publications mathématiques de l'I.H.E.S., $33, 21-57, 1967$

For a "good" map f, and a given non-increasing sequence $I = (n_1, \ldots, n_k)$ (called the **Boardman's symbol**), if $\Sigma^I(f)$ is a subvariety, then

$$\Sigma^{n_1,\ldots,n_k,n_{k+1}}(f) := \Sigma^{n_{k+1}}(f_{|\Sigma^I(f)})$$

is well-defined.

4.3 – Thom-Boardman's flags

In our case, one obtains the chain of inclusions :

$$\mathbb{C}^n \supseteq \Sigma^{n_1}(f) \supseteq \Sigma^{n_1,n_2}(f) \supseteq \ldots \supseteq \Sigma^{n_1,\ldots,n_k}(f)$$

and thus :

$$T_\zeta\mathbb{C}^n \supseteq T_\zeta\Sigma^{n_1}(f) \supseteq T_\zeta\Sigma^{n_1,n_2}(f) \supseteq \ldots \supseteq T_\zeta\Sigma^{n_1,\ldots,n_k}(f)$$

This suggests the following definitions

$$\begin{aligned} K_1(\zeta) &= \ker Df(\zeta) \\ K_2(\zeta) &= K_1(\zeta) \cap T_\zeta\Sigma^{n_1}(f) \\ \vdots &= \vdots \\ K_{k+1}(\zeta) &= K_1(\zeta) \cap T_\zeta\Sigma^{n_1,\ldots,n_k}(f) \end{aligned}$$

which are a particular case of our main construction.

5 – The main construction

5.1 – Intrinsic derivatives

Construction initiated by Porteous (1971), reconcile Rall's pioneer ideas and Thom-Boardman's stratification.
Yongjian Xiang (1998) gives regular defining equations for Thom-Boardman strata and define augmented systems.
Main idea : Construct equivariant differential operators at each order.

finition 2 *A* **reparametrization** *of f is the result of a changing of some coordinates (by analytic diffeomorphisms) in the source and in the target space.*

$$Diff(\mathbb{C}^n,\zeta) \times Diff(\mathbb{C}^n,0) \times \mathbb{C}\{z_{1:n}\} \rightarrow \mathbb{C}\{z_{1:n}\}$$
$$((\phi,\psi),f) \rightarrow (\phi,\psi).f := \psi \circ f \circ \phi^{-1}$$

Let us fix (ϕ,ψ) and denote by $\widetilde{f} := (\phi,\psi).f$.
If ζ is a zero of f, then comes immediately

$$D(\widetilde{f})(\zeta) = D(\psi)(0)Df(\zeta)D(\phi^{-1})(\zeta)$$

Let i_1 (resp. $\widetilde{i_1}$) be the canonical inclusion of $K_1(\zeta) = \ker Df(\zeta)$ (resp. $\widetilde{K_1(\zeta)} := \ker D\widetilde{f}(\zeta)$) and resp. $\widetilde{p_1}$) be the orthogonal projection onto the cokernel $L_1(\zeta) = cokerDf(\zeta) := T_0\mathbb{C}^n\mathrm{im}Df(\zeta)$ (resp. $(\zeta) := cokerD\widetilde{f}(\zeta)$), the following equality holds :

$$
\begin{aligned}
D^2\widetilde{f}(\zeta)(z-\zeta) &= D(D\psi\ Df\ D\varphi^{-1})(\zeta)(z-\zeta) \\
&= D^2\psi(\zeta)(z-\zeta)\ Df(\zeta)\ D\varphi(\zeta)^{-1} \\
&\quad + D\psi(\zeta)\ D^2 f(\zeta)(z-\zeta)\ D\varphi(\zeta)^{-1} \\
&\quad + D\psi(\zeta)\ Df(\zeta)\ D(D\varphi^{-1})(\zeta)(z-\zeta)
\end{aligned}
$$

Now observe that, when restricting to the kernel $\widetilde{K_1}(\zeta)$ and projecting onto the cokernel $\widetilde{L_1}(\zeta)$, the following equality holds

$$
\begin{aligned}
\widetilde{p_1} \circ D^2\widetilde{f}(\zeta)(z-\zeta) \circ \widetilde{i_1} &= \widetilde{p_1} \circ D\psi(\zeta)\ D^2 f(\zeta)(x-\zeta)\ D\varphi(\zeta)^{-1} \circ \widetilde{i_1} \\
&= D\psi(\zeta)\ (p_1 \circ D^2 f(\zeta)(x-\zeta) \circ i_1)\ D\varphi(\zeta)^{-1}
\end{aligned}
$$

The **first intrinsic derivative**, briefly defined by

$$
\delta_1(Df)(\zeta) := D(p_1 \circ Df \circ i_1)(\zeta) : T_\zeta\mathbb{C}^n \to T_0 Hom(K_1(\zeta), L_1(\zeta))
$$

is equivariant with respect to the previous group action. It induces a symetric bilinear operator

$$\delta_1^2 f(\zeta) : K_1(\zeta) \odot K_1(\zeta) \to L_1(\zeta)$$

The *restriction and projection* step is defined in local coordinates by taking the **Shur's complement** of the regular part of $Df(z)$.

For the definition of the second intrinsic derivative, we need $K_2(\zeta) := K_1(\zeta) \cap \ker \delta_1(Df)(\zeta) = \ker \delta_1^2 f$ and also $L_2(\zeta) := \mathrm{coker}(\delta_1^2 f(\zeta))$, with i_2 and p_2 the corresponding inclusion and projection, then

$$\delta_2(\delta_1^2 f)(\zeta) := \delta_1(p_2 \circ \delta_1^2 f \circ i_2)(\zeta)$$

The construction extends inductively.

References

I. R. Porteous, *The Normal Singularities of a Submanifold*, Journal of Differential Geometry $5, 543-564, 1971$

Yongjian Xiang, *Computing Thom-Boardman singularities*, Cornell University, Dr. Philosophy Thesis, 1998

5.2 – Intrinsic flags

$$
\begin{array}{rclcl}
K_1(\zeta) & = & \ker Df(\zeta) & & \\
K_2(\zeta) & = & K_1(\zeta) \cap \ker \delta_1(Df)(\zeta) & = & \ker \delta_1^2 f(\zeta) \\
K_3(\zeta) & = & K_2(\zeta) \cap \ker \delta_2(\delta_1^2 f)(\zeta) & = & \ker \delta_2^3 f(\zeta) \\
\vdots & = & \vdots & = & \vdots \\
K_{i+1}(\zeta) & = & K_i(\zeta) \cap \ker \delta_i(\delta_{i-1}^i f)(\zeta) & = & \ker \delta_i^{i+1} f(\zeta)
\end{array}
$$

Lecerf's example :

$$f_1 = x_1 + x_1^2 + x_2 + x_2^2 + 1/2x_3^2 - 1/2$$
$$f_2 = (x_1 + x_2 - x_3 - 1)^3 - x_1^3$$
$$f_3 = (1/5x_1^3 + 1/2x_2^2 + x_3 + 1/2x_3^2 + 1/2)^3 - x_1^5$$

$= (0, 0, -1)$ isolated root of multiplicity 18.

$$K_1(\zeta) = \{x_1 + x_2 - x_3 = 0\} \qquad n_1 = 2$$
$$K_2(\zeta) = K_1(\zeta) \qquad n_2 = 2$$
$$K_3(\zeta) = K_2(\zeta) \cap \{x_2 - x_3 = 0\} \qquad n_3 = 1$$
$$K_4(\zeta) = K_3(\zeta) \qquad n_4 = 1$$
$$K_5(\zeta) = K_4(\zeta) \qquad n_5 = 1$$
$$K_6(\zeta) = K_5(\zeta) \cap \{x_3 = 0\} = \{0\} \qquad n_6 = 0$$

denote it by $\zeta \in \Sigma^{2,2,1,1,1,0}(f)$

5.3 – Genericity conditions simplified

An other advantage : the genericy conditions given by Boardman with the sophistication of infinitesimal structures can be expressed in terms of intrinsic derivatives :

Proposition 1 *Suppose $\zeta \in \Sigma^{n_1,\ldots,n_k}(f)$, then f is $(n_1,\ldots,n_k)$-generic iff all its intrinsic derivatives up to order k*

$$\delta_1(Df)(\zeta)\,,\ldots,\;\delta_k(\ldots(\delta_1(Df))\ldots)(\zeta)$$

are surjective.

One recovers Morin's result which states that the generalized Whitney's maps are generic.

6 – Main result

inition 3 *Let us define*

$$\gamma_0 := \gamma(f, Df(\zeta), \zeta) = \max\left(1, \sup_{k\geq 2}\left(\frac{\|Df(\zeta)^\dagger D^k f(\zeta)\|}{k!}\right)^{1/(k-1)}\right)$$

l the following intrinsic point estimates

$$\gamma_i := \gamma_i^{int}(f,\zeta) = \max\left(1,\ \sup_{k\geq i+2}\left(\frac{\max_v \|\left(\delta_i^{i+1} f(\zeta) v^i\right)^\dagger \delta_i^k f(\zeta)\|}{k!}\right)^{1/(k-i-1)}\right)$$

en v runs over the unit sphere of $K_i(\zeta)$.

mme 1

$$\|\varepsilon_0 - \varepsilon_1\| = \mathcal{O}(\|\varepsilon_0\|)$$

orem 2 (intrinsic γ-theorem) *Let z_0 be a random point in the open polydisk $\Delta_0 = \{\|z - \zeta\| < 1/\gamma_0\}$, pose moreover that, for every i such that $n_i > 0$, the projection $\pi_i(z_0)$ belongs to $= \{\|pi_i(z) - \pi_i(\zeta)\| < 1/\gamma_i\}$, then*

$$|\pi_i^\perp \left(\varepsilon_1 - \left(\frac{i-1}{i}\right) \varepsilon_0\right)\| \leqslant \frac{(i+1)\left(\gamma_{i-1}\|\pi_{i-1}(\varepsilon_0)\|\right) - i\left(\gamma_{i-1}\|\pi_{i-1}(\varepsilon_0)\|\right)^2}{\left(1 - \left(\gamma_{i-1}\|\pi_{i-1}(\varepsilon_0)\|\right)\right)^2} \|\pi_{i-1}(\varepsilon_1 - \varepsilon_0)\| + \frac{\left(\gamma_{i-1}\|\pi_{i-1}(\varepsilon_0)\|\right)}{1 - \left(\gamma_{i-1}\|\pi_{i-1}(\varepsilon_0)\|\right)} \|\pi_{i-1}(\varepsilon_0)\|$$

l

$$\|\pi_i^\perp \left(\varepsilon_1 - \left(\frac{i-1}{i}\right) \varepsilon_0\right)\| = \mathcal{O}(\|\varepsilon_0\|^2)$$

demonstration is based on the Majorant series technique.

ollary 1 *If z_0 is as above, then the corrected sequence $(y_k)_{k\geqslant 1}$ defined by*

$$y_k = (\pi_1^\perp(z_k), \pi_2^\perp(2z_k - z_{k-1}), \ldots, \pi_{\mu-1}(\mu z_k - (\mu - 1)z_{k-1}))$$

verge quadratically to ζ.

non generic $\Sigma^{2,1}$:

$$f_1 = x_1 + x_2 - x_3$$
$$f_2 = x_1^2 + x_2^3 + x_3^3$$
$$f_3 = x_1 x_2 x_3$$

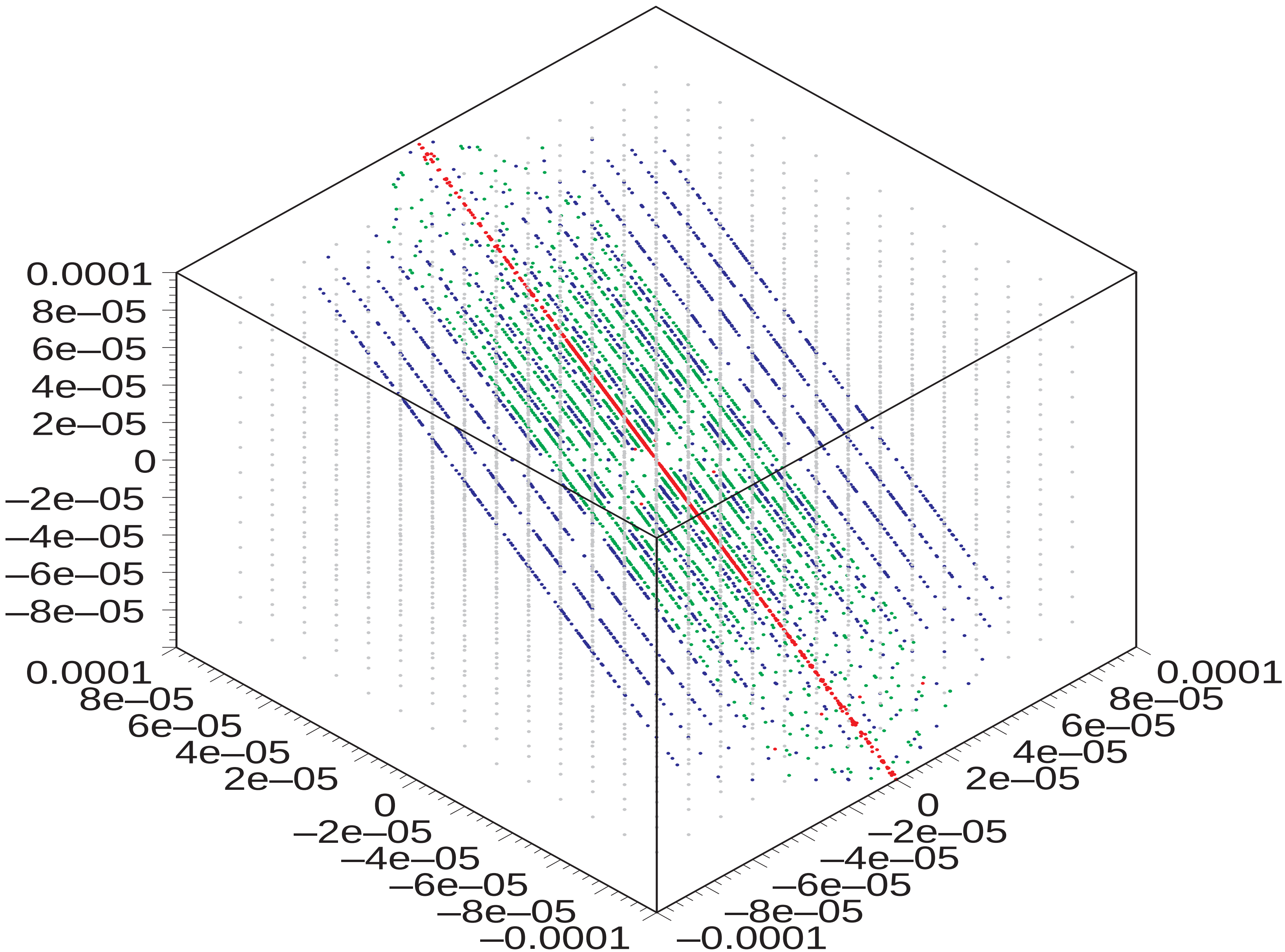
$0,0)$ is a singular zero of multiplicity 7 (SINGULAR)

$$Df(x) = \begin{pmatrix} 1 & 1 & -1 \\ 2x_1 & 3x_2^2 & 3x_3^2 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{pmatrix}$$

$$\delta_1^2 f(x) = \begin{pmatrix} 6x_2+2 & -2 & -2 & 6x_3+2 \\ -2x_3 & 2x_3-2x_2 & 2x_3-2x_2 & 2x_2 \end{pmatrix}$$

$$\delta_2^3 f(x) = \begin{pmatrix} \frac{-2}{3x_3+1} + \frac{6x_3}{(3x_3+1)^2} & \frac{-2}{3x_3+1} + \frac{6x_3}{(3x_3+1)^2} & 2 + \frac{6x_3+2}{3x_3+1} - \frac{3x_3(6x_3+2)}{(3x_3+1)^2} + \frac{6x_3}{3x_3+1} \end{pmatrix}$$

$$K_1(0) = \{x_1 = x_3 - x_2\} \supseteq K_2(0) = \{x_1 = 0, x_2 = x_3\} \supseteq K_3(0) = \{0\}$$


0.0001
8e–05
6e–05
4e–05
2e–05
0
–2e–05
–4e–05
–6e–05
–8e–05
0.0001
8e–05
6e–05
4e–05
2e–05
0
–2e–05
–4e–05
–6e–05
–8e–05
–0.0001
0.0001
8e–05
6e–05
4e–05
2e–05
0
–2e–05
–4e–05
–6e–05
–8e–05
–0.0001

7 – Geometric-Numeric computation of the Boardman symbol

e f is supposed analytic over a connected open $\mathcal{U} \subset \mathbb{C}^n$ with only one isolated root.

7.1 – One variable case

multiplicity of f at ζ can be obtain by means of the Newton's iterates $\{z_k\}_{k \geqslant 0}$ with the ratio

$$\frac{|z_{k+1} - z_k|}{|z_{k+1} - z_{k-1}|} = \frac{\mu - 1}{\mu}$$

7.2 – n-variables case

s the knowledge of the first n Newton iterates provide the sequence $n_1 \geqslant \ldots \geqslant n_l > 0$?

n ingredients

When are the two vectors $z_j - z_0$ and $z_k - z_0$ **nearly colinear** ?

When $\frac{\|(z_j - z_0) \wedge (z_k - z_0)\|}{\|z_j - z0\| \|z_k - z0\|} < \rho^2$ where ρ denotes the radius of the current open ball.

Determination of the Least Square Affine Subspace

$$\min_{(a_0, a_1, \ldots, a_{n-1})} \Sigma_{i=1}^{n} z_n^i - a_{n-1} z_{n-1}^i - \ldots - a_1 z_1^i - a_0$$

involves Gauss Pivot method.

Orthogonal projections

z_0 be a random point in the open polydisk $\Delta_0 = \{\|z - \zeta\| < 1/\gamma_0\}$ and set
$_{1} = N_f(z_k),\ 0 \leqslant k \leqslant n-1.$

orithm

input : $z_0, \ldots, z_n$

begin $i := 1;\ d := n\ s :=$ EMPTY STRING

while $d > 1$ do

- make the correction $z_{k+1} := z_{k+1} - \left(\frac{i-1}{i}\right) z_k,\ 0 \leqslant k \leqslant n-1$
- determine the dimension of the Least Square Affine Subspace (LSAS) and refresh d with the current dimension
- determine the equation of the LSAS by resolving the minimizatoin problem
- replace $z_1, \ldots, z_n$ by their projections onto the LSAS
- compute the next i for which $\frac{\|z_1 - z_2\|}{\|z_1 - z_0\|} = \frac{i-1}{i}$ and complete the sequence with the right occurence of d

end

output : $s = n_1, \ldots, n_l$

8 – Application to bifurcation problems

nsider the non linear differential system

$$\partial_t X(t) = f(X(t), \lambda)$$

re

- $f : \mathcal{X} \times \mathbb{K}^p \longrightarrow \mathcal{Y}$ between two Banach spaces,
- X is the **state variable**, lying in a Banach space ($\mathcal{X} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ or $\mathbb{C}\{z\}^n$),
- λ (in $\mathbb{K}^p = \mathbb{R}^p$ or $\mathbb{C}^p$) is the **bifurcation parameter**
- **Motivation :** Study of the possible bifurcations (topological changes in the phase portrait) of equilibrium solution $f(X_0, \lambda_0) = 0$, especially if it is a singular point of f.

8.1 – Reduction step

aim to obtain a finite dimensional problem **qualitatively similar** (type and unfolding of the singular point).

- **Lyapunov-Schmidt reduction** (drawback : it requires the knowledge of $K_1 = Ker(D_X f(X_0, \lambda_0))$ and $R_1 = Im(D_X f(X_0, \lambda_0))$),
- **Generalized Lyapunov-Schmidt method** provides numerical approximations of K_1 and R_1.

A. D. Jepson, A. Spence *On a reduction process for nonlinear equations*, SIAM J. Math. Anal., Vol. 20, No. 1, January 1989

punov-Schmidt reduction

$f(X,\lambda)=0,\ D_X f(X_0,\lambda_0)$ is Fredholm of index 0 $(dim(K_1)=codim(R_1))$

$\mathcal{X}=K_1\oplus M$

$\mathcal{Y}=R_1\oplus N$

$IFT\Rightarrow \pi_{R_1}\circ f(\pi_{K_1}(X)+\theta(\pi_{K_1}(X),\lambda),\lambda)\equiv 0$

For $X_1\in K_1,$ define $\varphi(X_1,\lambda):=(id-\pi_{K_1})fk(X_1+\theta(X_1,\lambda),\lambda)$

Fix $K_1=Span\{v_1,\ldots,v_{n_1}\}$ and $R_1^{\perp}=Span\{v_1^*,\ldots,v_{n_1}^*\}$

and define $g=(g_1,\ldots,g_{n_1})$ by setting

$g_i(x,\lambda)=<v_i^*,\varphi(x_1v_1+\ldots+x_{n_1}v_{n_1},\lambda)>$

Lyapunov-Schmidt theorem relates the initial problem to the determination of the type of singular point of the uced system we are dealing with.

8.2 – Geometrical aspect

Two dynamical systems have the same qualitative behavior iff their reduced systems are **contact equivalent** (in the sense of Golubitsky and Schaeffer), therefore, iff they have the same geometry at the singular point.

Golubitsky and Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. I, Springer-Verlag, 1985

In the case of a finite dimensional local algebra, we have seen that the behaviour of the Newton process is very informative!

8.3 – Numerical experiment

The reaction-diffusion model called *the Brusselator* presents Hopf and Pitchfork bifurcations.

(joint work with Ali Faraj, INSA TOULOUSE)

W. Govaerts *Computation of singularities in large nonlinear systems*, SIAM J. Numer. Anal., Vol. 34, No. 3, June 1997

Thanks for your invitation.