Limit distributions and scaling behaviour for models of planar polygons

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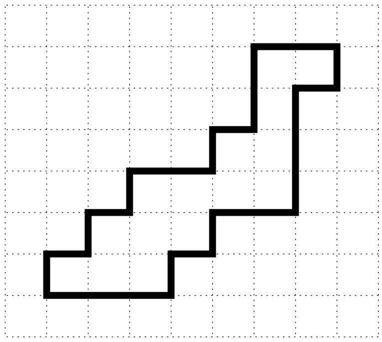
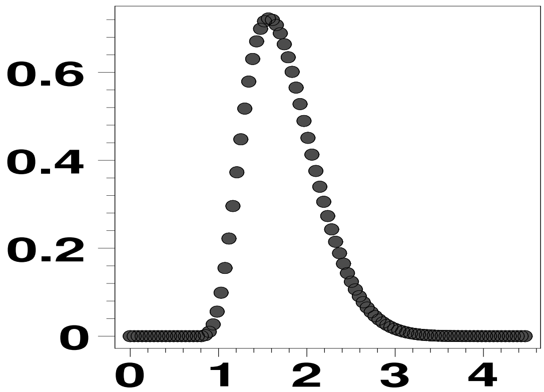
- Limit distributions
- q-algebraic functional equations
- formal asymptotic expansions
- self-avoiding polygons
- planar random loop boundary
- summary & outlook

(T. Guttmann, I. Jensen, T. Prellberg)

Staircase polygons: limit distribution



staircase polygons



area distribution

$p_{m,n}$ #polygons of half-perimeter m, area n

area distribution: $\mathbb{P}(X_m = n) = \frac{p_{m,n}}{\sum_n p_{m,n}}$

Mean area: $\mathbb{E}[X_m] = \mathcal{O}(m^{3/2})$

Limit $m \to \infty$: $\mathbb{P}[\frac{4X_m}{m^{3/2}} \leq x] \to \mathbb{P}[X \leq x]$ (point-wise), where X is Airy distributed.

$$\mathbb{E}[X^k] = -\frac{\Gamma(-1/2)}{\Gamma((3k-1)/2)}\Omega_k$$

$$\frac{d}{dz}\log \mathrm{Ai}(z) = \sum_{k\geq 0} \frac{(-1)^k}{k!}\frac{\Omega_k}{2^k} z^{-(3k-1)/2}$$

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + tz\right) dt$$

(compare Flajolet et al. 1998, Duchon 1999)

Limit distributions: generating function approach

$p_{m,n}$ #polygons of half-perimeter m, area n

Select polygons uniformly among all polygons with fixed perimeter m:

$$\mathbb{P}(X_m = n) = \frac{p_{m,n}}{\sum_n p_{m,n}}$$

Expectation value of moments of X_m:

$$\mathbb{E}[X_m^k] = \frac{\sum_n n^k p_{m,n}}{\sum_n p_{m,n}}$$

Perimeter & area generating function:

$$G(x,q) = \sum_{m,n} p_{m,n} x^m q^n$$

Generating function approach:

Express (asymptotic) behaviour of $\mathbb{E}[X_m^k]$ in terms of (singular) behaviour of $G(x,q)$.

Generating functions and area moments

Factorial area moment generating functions

$$g_k(x) = \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}q^k} G(x,q)\bigg|_{q=1}$$
$$= \frac{(-1)^k}{k!} \sum_m \left(\sum_n (n)_k \, p_{m,n} \right) x^m$$

$(a)_k = a(a-1)\cdots(a-k+1)$

Area distribution moments...

$$\mathbb{E}[X_m^k] \sim \mathbb{E}[(X_m)_k] = (-1)^k k! \frac{[x^m]g_k(x)}{[x^m]g_0(x)}$$

...determined by singular behaviour of $g_k(x)$

$$g_k^{(sing)}(x) \sim \frac{f_k}{(x_c - x)^{\gamma_k}}$$
$$[x^m]g_k(x) \sim \frac{f_k}{x_c^{\gamma_k}\Gamma(\gamma_k)} x_c^{-m} m^{\gamma_k - 1}$$

Generating functions and area moments

Assume exponents $\gamma_k = (k-\theta)/\phi$

"critical exponents" θ, ϕ

Define "distribution coefficients" Φ_k via

$$f_k = -\frac{(-1)^k}{k!} 2^k \Phi_k f_1^k f_0^{1-k}$$

Expectation value of area moments

$$\mathbb{E}[X_m^k] \sim -\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} \left(\left(\frac{m}{x_c} \right)^{1/\phi} \frac{2 f_1}{f_0} \right)^k \Phi_k$$

Introduce normalised random variable $\widetilde{X}_m$:

$$\widetilde{X}_m = \frac{X_m}{\left(\frac{m}{x_c}\right)^{1/\phi} \frac{2f_1}{f_0}}$$

Expectation value of moments of $\widetilde{X}_m$:

$$\mathbb{E}[\widetilde{X}_m^k] \sim -\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} \Phi_k$$

constants Φ_k, θ, ϕ characterise limit distribution

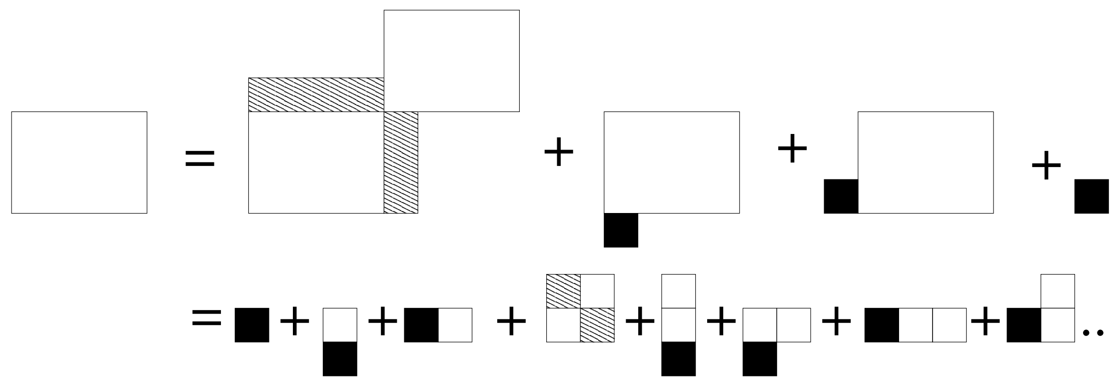
Staircase polygons: exact solution

Functional equation for perimeter & area generating function

$$\begin{aligned} G(x,q) &= G(qx,q)G(x,q) + 2qxG(x,q) + qx^2 \\ &= qx^2 + 2q^2x^3 + (q^4 + 4q^3)x^4 + \ldots \end{aligned}$$

"q-quadratic functional equation of first order"

Diagrammatic representation



Linearisation via Ansatz $G(x,q) = qx^2 \frac{L(qx,q)}{L(x,q)}$

$$L(x,q) = \sum_{n,m>0} \frac{(-x)^{n+m} q^{\binom{n+m+1}{2}}}{(q;q)_n (q;q)_m}$$

q-deformed Bessel function $L(x,q)$

q-product $(t;q)_n = \prod_{k=0}^{n-1}(1-q^k t)$

Explicit solution in terms of q-series

(e.g. Prellberg, Brak 1995)

Extension: q-algebraic functional equations

$$P(G(x,q), G(qx,q), \ldots, G(q^N x,q), x, q) = 0$$

$P(y_0, y_1, \ldots, y_N, x, q)$ polynomial in its variables

Interpretation:
Polygons satisfy a recursion of finite depth N

Limit $q \to 1$ leads to *algebraic differential equation* for perimeter generating function $G(x,1)$

Simplest case: perimeter generating function algebraic

$$P(G(x,1), G(x,1), \ldots, G(x,1), x, 1) = 0$$

(applies to many exactly solved polygon models)

Staircase polygons: $G(x,1) = \frac{1}{2} - x - \frac{1}{2}\sqrt{1-4x}$

Singular behaviour about a square-root

Assume algebraic perimeter generating function $G(x,1)$ with square-root singularity:

$$\left(\sum_k \partial_k P\right) = 0, \quad \left(\sum_{k,l} \partial_{k,l} P\right) \neq 0, \quad (\partial_x P) \neq 0$$

$\partial_x = \frac{\partial}{\partial_x}$, $\partial_k = \frac{\partial}{\partial_{y_k}}$, argument $(x,q) = (x_c, 1)$

Assume in addition $(\sum_k k \partial_k P) \neq 0$

Puiseux expansion for factorial area moment generating functions:

$$g_k(x) = \sum_{l=0}^{\infty} f_{k,l} (x_c - x)^{-\frac{3}{2}k + \frac{(l+1)}{2}}$$

$$f_{k,0} = \frac{(-1)^k}{k!} 2^k \Phi_k f_{1,0}^k f_{0,0}^{1-k}$$

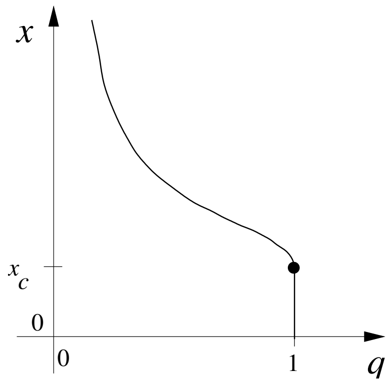
$$f_{0,0}^2 = \frac{(\partial_x P)}{\frac{1}{2}\left(\sum_{k,l} \partial_{k,l} P\right)} \qquad -4 f_{1,0} = x_c \frac{\left(\sum_k k \partial_k P\right)}{\frac{1}{2}\left(\sum_{k,l} \partial_{k,l} P\right)}$$

Airy distribution coefficients: $\Phi_k = \Omega_k$

(Duchon 1999, R2002)

Corrections to asymptotic behaviour?

Phase diagram of staircase polygons: Radius of convergence of $G(x,q)$



$q=1$: finite essential singularity (extended phase)

$q<1$: simple pole (collapsed phase)

Scaling behaviour about $(x,q)=(x_c,1)$

$$G^{(sing)}(x,q) \sim (1-q)^{\theta} F_0((x_c-x)(1-q)^{-\phi})$$

as $x \to x_c = 1/4$, uniformly in q (Prellberg 1995)

Scaling function $F_0(z)$ for staircase polygons:

$$F_0(z) = \frac{1}{16}\frac{d}{dz}\log \mathrm{Ai}(2^{\frac{8}{3}}z), \; (\theta,\phi)=(\tfrac{1}{3},\tfrac{2}{3})$$

Asymptotic expansion (about $z=\infty$) $\to$ area moments

Area moments from the scaling function

Singular behaviour of factorial area moment generating functions...

$$g_k^{(sing)}(x) \sim \frac{f_k}{(x_c - x)^{\gamma_k}},$$

where $\gamma_k = (k - \theta)/\phi$

...appears in asymptotic expansion of the scaling function

$$F_0(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^{\gamma_k}}$$

Alternative method to determine f_k:

Extract $F_0(z)$ directly from the functional equation and compute asymptotic expansion of $F_0(z)$

Area moments from the scaling function

Assume scaling behaviour

$$G^{(sing)}(x,q) \sim (1-q)^{\theta} F_0\left(\frac{x_c - x}{(1-q)^{\phi}}\right)$$

Assume that $F_0(z) = \sum_k a_k / z^{\alpha_k}$, where $a_k \neq 0$, $\alpha_{l+1} > \alpha_l$, and $\phi > 0$.

Differentiate:

$$\frac{(-1)^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}q^k} G^{(sing)}(x,q) \sim \sum_{l=0}^{\infty} \frac{a_l}{(x_c - x)^{\alpha_l}} \frac{(\theta + \alpha_l \phi)_k}{k!} (1-q)^{-k+(\theta+\alpha_l\phi)}$$

Finite limit as $q \to 1$:
$\theta + \alpha_{l_k}\phi = k$ for some integer $l_k \geq 0$ and
$(\theta + \alpha_l \phi)_k = 0$ for $0 \leq l < l_k$

Conclusion: $l_k = k$

Compare to asymptotic form of $g_k^{(sing)}(x)$:
$\alpha_k = \gamma_k$ and $a_k = f_k$

Thus

$$F_0(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^{\gamma_k}}, \qquad \gamma_k = \frac{k-\theta}{\phi}$$

Scaling function from the q-algebraic functional equation

Introduce $z = (x_c - x)(1-q)^{-\phi}$ and take limit $q \to 1$ in the functional equation, using the Ansatz

$$G^{(sing)}(x,q) \sim (1-q)^{\theta} F_0((x_c - x)(1-q)^{-\phi})$$

(Non-linear) differential equation for scaling function $F_0(z)$!

Square-root singularity ($\theta = 1/3$, $\phi = 2/3$)

$$F_0(z)^2 - 4 f_{1,0} F_0'(z) - f_{0,0}^2 z = 0$$

with constants $f_{1,0}, f_{0,0} \neq 0$ (see above)

Riccati equation with solution:

$$F_0(z) = -4 f_{1,0} \frac{\mathrm{d}}{\mathrm{d}z} \log \mathrm{Ai}\left(\left(\frac{f_{0,0}}{4 f_{1,0}}\right)^{2/3} z\right)$$

Extension: Correction-to-scaling functions

Assume Puiseux expansion of factorial area moment generating functions

$$g_k(x) = \sum_{l=0}^{\infty} \frac{f_{k,l}}{(x_c - x)^{\gamma_{k,l}}}$$

with exponents $\gamma_{k,l} = (k - \theta_l)/\phi$

Consistent with asymptotic expansion of the perimeter and area generating function

$$G^{(sing)}(x, q) \sim \sum_{l=0}^{\infty} (1-q)^{\theta_l} F_l \left(\frac{x_c - x}{(1-q)^{\phi}} \right)$$

with correction-to-scaling functions

$$F_l(z) = \sum_{k=0}^{\infty} \frac{f_{k,l}}{z^{\gamma_{k,l}}}$$

Method to determine $f_{k,l}$:

Extract differential equations for $F_l(z)$ directly from the functional equation and compute asymptotic expansion of $F_l(z)$.

Staircase polygons:
Correction-to-scaling functions

Ansatz for perimeter & area generating function

$$G(x,q) = G(x_c,1) + \sum_{l=0}^{\infty} (1-q)^{\frac{l+1}{3}} F_l\left(\frac{x_c - x}{(1-q)^{2/3}}\right)$$

$$F_0(z) = \frac{1}{16}\frac{\mathrm{d}}{\mathrm{d}z} \log \mathrm{Ai}\left(2^{\frac{8}{3}} z\right)$$

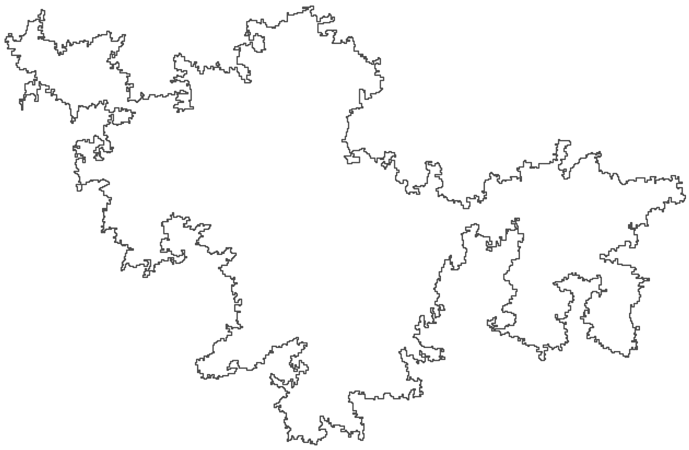
Functional equation yields inhomogeneous linear DE's for correction–to–scaling functions $F_l(z)$

$$\begin{aligned}
F_1(z) &= 0 \\
F_2(z) &= \frac{3}{40} - \frac{16}{15} z F_0(z) + \frac{32}{15} z^2 F_0'(z) \\
F_3(z) &= \frac{1}{6} F_0(z) - \frac{1}{3} z F_0'(z) \\
F_4(z) &= -\frac{89}{350} z - \frac{1936}{1575} z^2 F_0(z) \\
&\quad + \left(\frac{1888}{525} z^3 + \frac{9}{2240}\right) F_0'(z) \\
&\quad + \frac{512}{225} z^4 F_0''(z)
\end{aligned}$$

(compare Flajolet et al. 2002)

Asymptotic expansion of $F_l(z)$ yields $f_{k,l}$ in Puiseux expansion of $g_k(x)$

Self-avoiding polygons



Numerical observation: Area is Airy distributed!

moment extrapolation using exact enumeration data (R, Guttmann, Jensen 2001)

Independent of underlying lattice

tested for square, triangular, hexagonal lattice

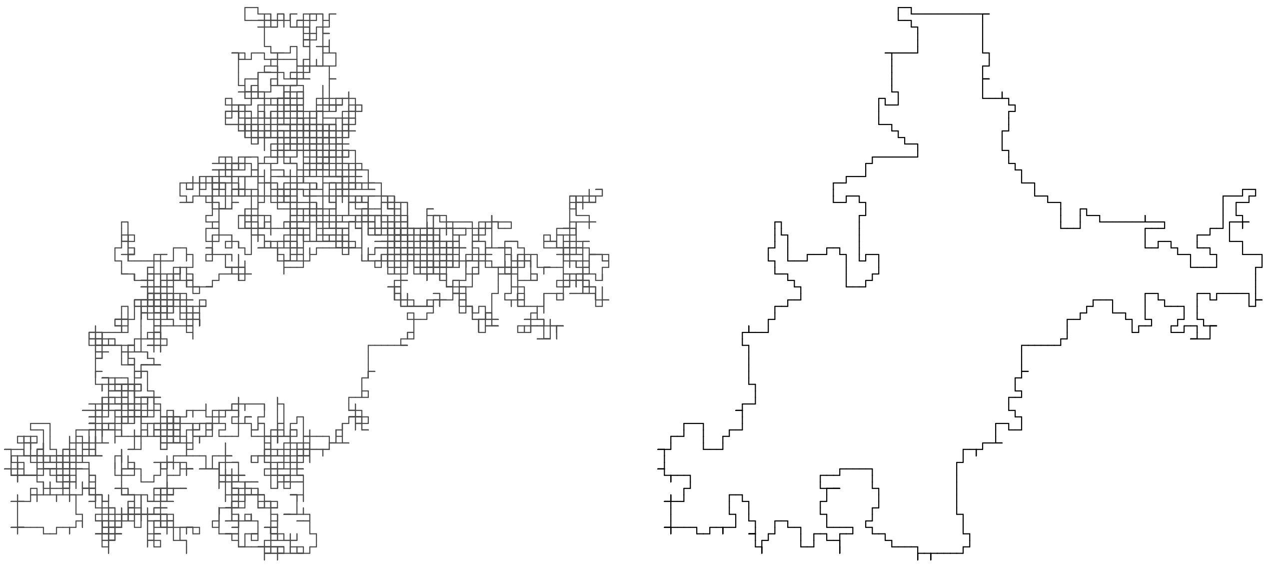
Reason?

Rooted SAPs $G^{(r)}(x,q) = \frac{d}{dx}G(x,q)$ display square-root singularity – same critical exponents as for staircase polygons (Guttmann 1995)

Field theoretical derivation of scaling function (Cardy 2001)

Conformally invariant scaling limit (if it exists) is described by Brownian excursions (Lawler et al. 2002), which are Airy distributed (Louchard 1984)

Planar random loop boundary



Continuum limit:
Hausdorff dimension 4/3 of the boundary
(Mandelbrot 1982, Lawler et al. 2001)

Like staircase polygons (self-avoiding polygons)

Monte Carlo simulation of random loops

Numerical observation: Area is Airy distributed!

Independent of underlying lattice
tested for square, triangular lattices

Summary & open problems

Limit distributions for planar polygon models

derived from q-algebraic functional equation

Square-root singularity yields Airy distribution

(Duchon 1999)

Airy distribution appears elsewhere

linear probing hashing, connected graphs,...

(Flajolet et al. 1998, Flajolet et al. 2002)

self-avoiding polygons, planar random loop boundary

(Alternative) derivation by stochastic approach?

SLE

Other types of singularity can be treated

Analysis of corrections to asymptotic behaviour

Formal asymptotic expansion of p&a generating function

differential equation for (correction-to-) scaling functions

Asymptotic expansion?

Proof of existence using the functional equation

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