Algorithms for the Construction of the Minimal Telescopers

H.Q. Le

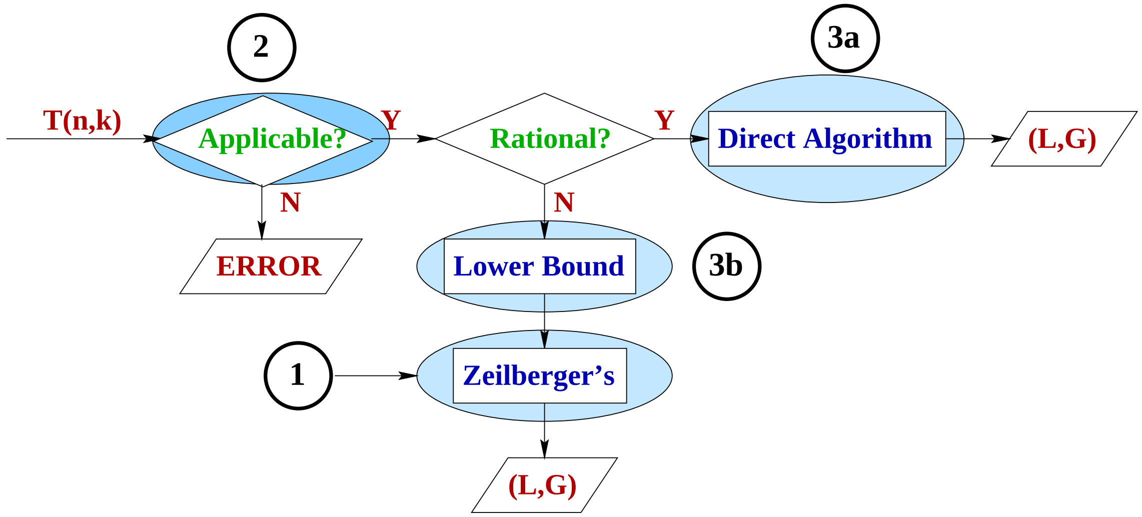
Algorithms Project

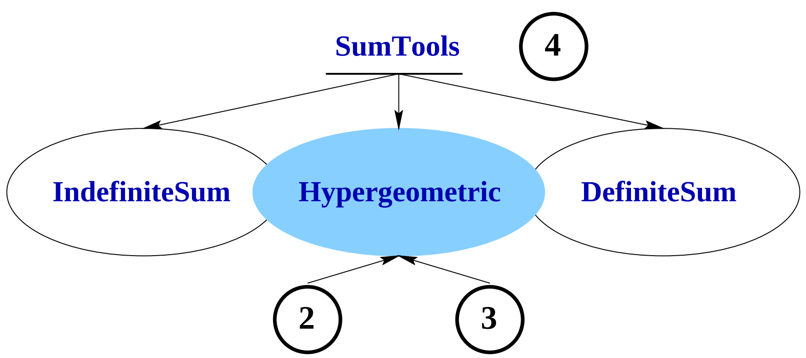
INRIA Rocquencourt

78153 Le Chesnay Cedex, France

Algorithms Project's Seminar – October 20, 2003

Summary


2
3a
T(n,k)
Applicable?
Y
Rational?
Y
Direct Algorithm
(L,G)
N
N
ERROR
Lower Bound
3b
1
Zeilberger's
(L,G)


SumTools
4
IndefiniteSum
Hypergeometric
DefiniteSum
2
3

Definitions and Notation

$\mathbb{K}$ ...... a field of characteristic 0

$T(k)$ ...... a sequence, or a function over $\mathbb{K}$

$E_k\, T(n,k) = T(n,k+1)$ ...... shift operator w.r.t. k

$E_n\, T(n,k) = T(n+1,k)$ ...... shift operator w.r.t. n

Creative Telescoping

input: a function $T(n, k)$.

output: a pair (L, G) where $L \in \mathbb{K}[n, E_n]$, i.e.,

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \; a_i \in \mathbb{K}[n],$$

and a function $G(n, k)$ such that

$$LT(n, k) = (E_k - 1)G(n, k).$$

Remark. The theory of creative telescoping was initially designed by Zeilberger (Zeilberger, 1991) for the case when the function $T(n, k)$ is a hypergeometric term. It was later adapted to holonomic functions (Chyzak & Salvy, 1998), (Chyzak, 2000).

Definitions and Notation

Definition. A sequence $T(k)$ over $\mathbb{K}$ is a hypergeometric term if there are $f, g \in \mathbb{K}[k] \setminus \{0\}$ such that

$$f(k)T(k+1) + g(k)T(k) \quad = \quad 0.$$

$\mathcal{C}_k(T) = T(k+1)/T(k) = -g(k)/f(k)$: the certificate of T.

Example 1.

$T(k)$	$\mathcal{C}_k(T) = \frac{E_k\, T(k)}{T(k)}$
2^k	2
$k!$	$k+1$
$\binom{2k}{k}$	$2\,\frac{2k+1}{k+1}$

Definition. A sequence $T(n,k)$ is a hypergeometric term if there are $f_0, f_1, g_0, g_1 \in \mathbb{K}[n,k] \setminus \{0\}$ such that

$$\begin{aligned} f_1(n,k)T(n+1,k) + f_0(n,k)T(n,k) &= 0, \\ g_1(n,k)T(n,k+1) + g_0(n,k)T(n,k) &= 0. \end{aligned}$$

$\mathcal{C}_n(T) = T(n+1,k)/T(n,k)$, $\mathcal{C}_k(T) = T(n,k+1)/T(n,k)$: the n-certificate and the k-certificate of T, respectively.

Zeilberger's Algorithm

D. Zeilberger. The method of creative telescoping. *Journal of Symbolic Computation* **11**, 195–204, 1991.

input : a hypergeometric term $T(n, k)$.

output: a Z-pair (L, G) where $L \in \mathbb{C}[n, E_n]$, i.e.,

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \ a_i \in \mathbb{C}[n],$$

and a hypergeometric term $G(n, k)$ such that

$$LT(n, k) = (E_k - 1)G(n, k).$$

Remark. The telescoper L in the returned Z-pair is of minimal possible order, and is called the minimal telescoper.

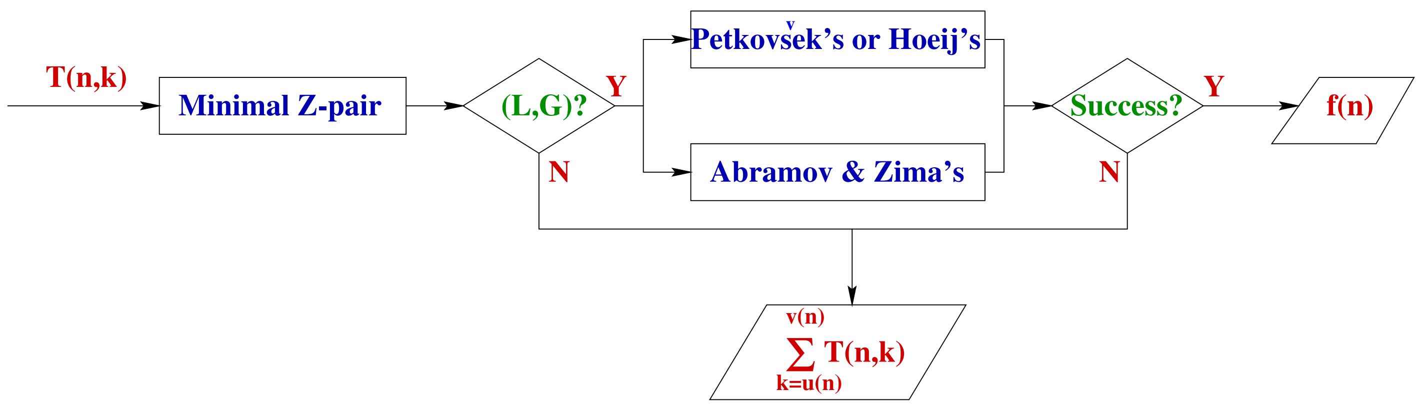
Applications

- finding closed forms of definite sums of hypergeometric terms ($\mathcal{ZP}$ method),
- certifying combinatorial identities ($\mathcal{WZ}$ method),
- proving transformation formulas for hypergeometric series (Paule, Strehl, 1995).

Closed Forms of Definite Sums

Goal: Compute a closed form of $f(n) = \sum_{k=u(n)}^{v(n)} T(n,k)$, where

$T(n,k)$ is a hypergeometric term in n and k.



Example 2 (Riordan, 1968)

$$f(n) = \sum_{k=0}^{n} T(n,k) = \sum_{k=0}^{n} \binom{2\,n}{2\,k}^2 .$$

```
> req := ZeilbergerRecurrence(T,n,k,f,0..n);
```

$$\begin{aligned} req := (10\,n^5 + 55\,n^4 + 112\,n^3 + 104\,n^2 + 43\,n + 6) f(n+2) \\ - (120\,n^5 + 600\,n^4 + 1124\,n^3 + 972\,n^2 + 376\,n + 48) f(n+1) \\ - (640\,n^5 + 2880\,n^4 + 4728\,n^3 + 3492\,n^2 + 1148\,n + 132) f(n) &= 0 \end{aligned}$$

```
> LREtools['hypergeomsols'](req,{f(n)},{f(0)=1,f(1)=2});
```

$$\sum_{k=0}^{n} \binom{2\,n}{2\,k}^2 = \frac{1}{2} \frac{4^n \left(\Gamma\left(2\,n + \frac{1}{2}\right)\sqrt{\pi} + (-1)^n \,\Gamma\left(n + \frac{1}{2}\right)^2\right)}{\sqrt{\pi}\Gamma\left(n + \frac{1}{2}\right)\Gamma(n+1)}$$

Zeilberger's Algorithm: the Strategy

The algorithm uses an item-by-item examination on the order ρ of L. It starts with the value of 0 for ρ, and increases ρ until it is successful in finding a Z-pair (L, G).

Example. Consider the hypergeometric term

$$T(n,k) = \frac{1}{n^2 + 9\,nk - 4\,n - 22\,k^2 + 21\,k - 5}.$$

```
> Zpair := Zeilberger(T,n,k,E_n):
  "applying Zeilberger's algorithm for order 0"
   ...
  "applying Zeilberger's algorithm for order 12"
> L := Zpair[1];
```

$$L = (13\,n + 157)E_n^{12} + (13\,n + 144)E_n^{11} - (13\,n + 14)E_n - (13\,n + 1)$$

Problem 1: Applicability of $\mathcal{Z}$

Problem. $\mathcal{Z}$ tries to compute a Z-pair for $T(n, k)$ when such a pair might not exist. In other words, $\mathcal{Z}$ might not terminate.

Side effect.

- An upper bound, e.g. 6, for the guessed order of a telescoper needs to be set in advance.
- It is not always possible to set the order of telescopers L to a "high enough" value. e.g.,

$$F_m(n, k) = \frac{1}{n + (m+1)\,k}, \quad m \in \mathbb{N}.$$

Additive Decomposition Problem (ADP)

input: a function $T(k)$ over $\mathbb{K}$.

output: two functions $T_1(k)$, $T_2(k)$ over $\mathbb{K}$ such that

$$T(k) = (E_k - 1)T_1(k) + T_2(k)$$

where $T_2(k)$ is "simpler" than T in some sense.

Remark. Any algorithm which solves the ADP should guarantee that if $T(k)$ is summable, then $T_2(k) \equiv 0$.

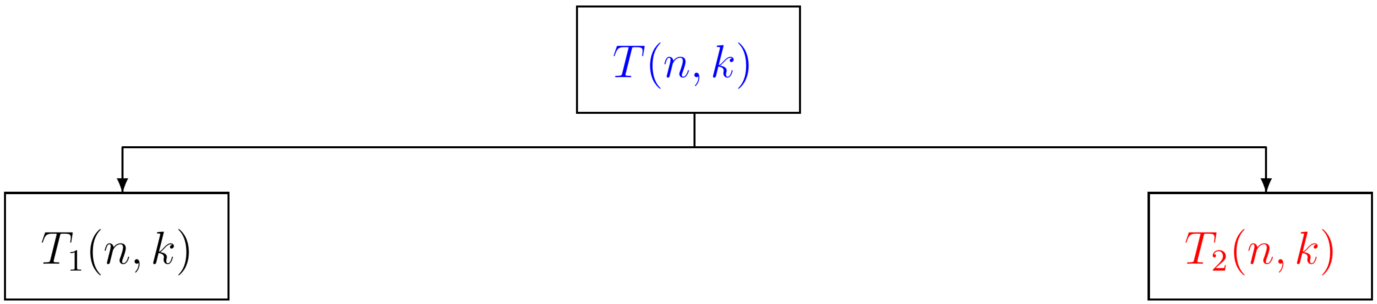
Applicability of $\mathcal{Z}$: the General Approach

For a given hypergeometric term $T(n, k)$, let $(T_1(n, k), T_2(n, k))$ be an additive decomposition of T w.r.t. k.

- If the non-summable part $T_2 \equiv 0$, then $\mathcal{Z}$ is obviously applicable to T.
- Otherwise, it is easy to show that $\mathcal{Z}$ is applicable to T iff it is applicable to T_2. Hence, the investigation will be on the structure of the non-summable part T_2.

Applicability of $\mathcal{Z}$ to a Rational Function

S.A. Abramov, H.Q. Le. A criterion for the applicability of Zeilberger's algorithm to rational functions. *Discrete Mathematics* **259**, 1–17, 2002.



$$T(n,k) = (E_k - 1)T_1(n,k) + T_2(n,k)$$

additive decomposition w.r.t. k

$$T_2(n,k) = \frac{f(n,k)}{g(n,k)}, \ f,\, g \in \mathbb{C}[n,k], \ \ (g \text{ of minimal degree w.r.t. } k)$$

$\mathcal{Z}$ is applicable to $T(n,k) \Longleftrightarrow g(n,k) \in Z_{n,k}$

$g(n,k) = \prod_i (a_i\, n + b_i\, k + c_i), \ a_i,\, b_i \in \mathbb{Z}, \ c_i \in \mathbb{C}$

ADP: Rational Function Case

S. A. Abramov, Rational component of the solutions of a first-order linear recurrence relation with a rational right-hand side. *USSR Comput. Math. Phys.* Transl. from *Zh. vychisl. mat. mat. fyz.* **14**, 1975, 1035–1039.

Given $T(k) \in \mathbb{K}(k)$, construct a pair of rational functions over $\mathbb{K}$ $(T_1(k), T_2(k))$ such that

$$T(k) = (E_k - 1)T_1(k) + T_2(k)$$

where the denominator of T_2 has the minimal possible degree.

A Sketch of the Proof

$$T(n,k) = (E_k - 1)T_1(n,k) + \frac{f(n,k)}{g(n,k)},\ \ f, g \in \mathbb{K}[n,k].$$

If $\exists (L, G)$ for T_2, i.e., $LT_2 = (E_k - 1)G = \dfrac{a(n,k)}{b(n,k)},\ \ a,\ b \in \mathbb{C}[n,k]$.

Then

$$g(n,k) \text{ is shift-free w.r.t. } k,\ \ b(n,k) \text{ is spread w.r.t. } k. \qquad (1)$$

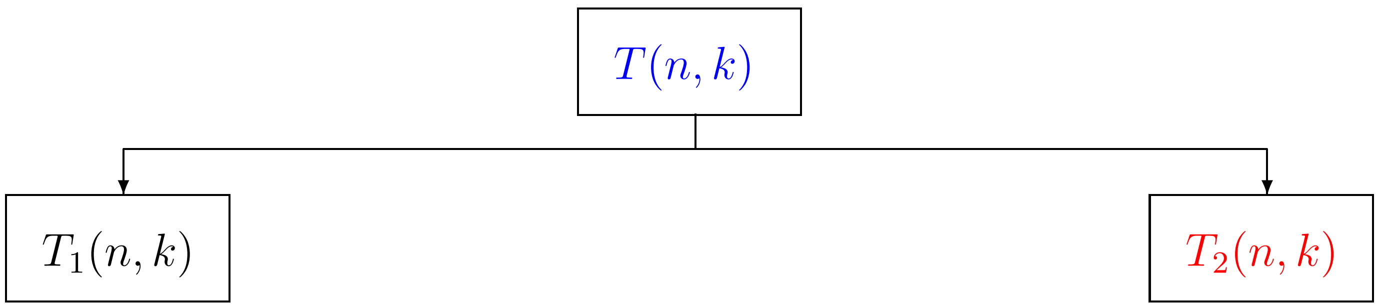
$$\overset{(1)}{\Longrightarrow} \quad \forall p_1 = (k - \alpha_i(n)),\ p_1 \,|\, g,\ \exists\, J,\ H \in \mathbb{Z},\ J > 0 : \qquad (2)$$

$$p_1(n,k) = p_1(n + J, k + H)$$

$$\overset{(2)}{\Longrightarrow} \quad p_1(n,k) = \tfrac{1}{J}\,(Jk - Hn + Jc),\ \ c \in \mathbb{C}$$

Applicability of $\mathcal{Z}$ to a Hypergeometric Term

S.A. Abramov. When does Zeilberger's algorithm succeed? *Advances in Applied Mathematics* **30** (2003) 424–441.



$$T(n,k) = (E_k - 1)T_1(n,k) + T_2(n,k)$$

additive decomposition w.r.t. k

$$(z, f_1, f_2, v_1, v_2) := \mathrm{RNF}_k(\mathcal{C}_k(T_2(n,k)))$$

$\mathcal{Z}$ is applicable to $T(n,k) \iff v_2(n,k) \in Z_{n,k}$

Example 3

$$T = \frac{nk}{(n+k-2)(nk+1)}\binom{n}{k} + \frac{1}{(nk+n+1)(n+k-1)}\binom{n}{k+1}.$$

An additive decomposition (T_1, T_2) of T w.r.t. k has

$$T_2 = \frac{33\,kn - 24\,k - 5\,kn^2 + 67\,n - 24 + 3\,n^3 - 22\,n^2}{24(n+k-1)} \prod_{w=0}^{k-1} \frac{n-w}{w+4}.$$

Since $v_2 \in Z_{n,k}$, $\mathcal{Z}$ is applicable to $T(n,k)$.

Example 4

$$T(n,k) = \frac{-2\,nk^2 + (n^2-1)k + n + 1}{nk^2 + (n+1)k + 1}\,\frac{\Gamma(n+1)}{\Gamma(n-k+2)\,\Gamma(k+1)}.$$

An additive decomposition (T_1, T_2) of T w.r.t. k yields

$$T_1 = \frac{2\,nk + 2\,n - 1}{2\,n\,(n+1)} \prod_{w=0}^{k-1} \frac{n-w+1}{w+2},$$

$$T_2 = \frac{(n^2+5\,n-2)k + 5\,n - 1}{4\,n\,(n+1)(nk+1)} \prod_{w=0}^{k-1} \frac{n-w+1}{w+3}.$$

Since $v_2 = 4\,n\,(n+1)(nk+1) \notin Z_{n,k}$, $\mathcal{Z}$ is not applicable to $T(n,k)$.

Factorization into Integer-Linear Polynomials

input : a polynomial $g(n,k) \in \mathbb{C}[n,k]$.

output: *true* if $g(n,k) \in Z_{n,k}$, *false* otherwise.

$$w(n,k) = \frac{g(n,k)}{\mathrm{content}_k\, g(n,k) \cdot \mathrm{content}_n\, g(n,k)}.$$

$$w(n,k) \stackrel{?}{=} \prod_{i=0}^{m} w_{c_i}(n,k),\ \ c_i \in \mathbb{Q} \setminus \{0\},\ c_i \neq c_j \text{ for } i \neq j.$$

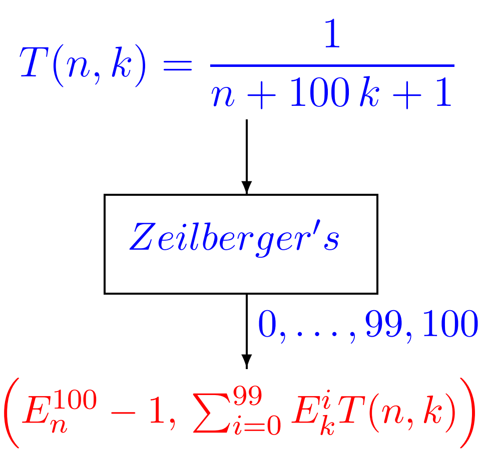
$$w_{c_i}(n,k) = \prod_j (k + c_i\, n + \gamma_j),\ \ \gamma_j \in \mathbb{C}.$$

Remark. This procedure does not require a complete factorization of the input polynomial $g(n,k)$ into irreducible factors.

Problem 2: Efficiency of $\mathcal{Z}$

Due to the item-by-item examination strategy, let ρ be the order of the minimal telescoper for the input hypergeometric term $T(n,k)$. Then $\mathcal{Z}$ wastes resources trying to compute a Z-pair for T where the guessed orders of the telescopers are less than ρ.

Example.



Rational Function Case: a Direct Algorithm

A Direct Algorithm to Construct the Minimal Z-pairs for Rational Functions. *Advances in Applied Mathematics*, 30, 137–159, 2003.

An efficient algorithm which computes a Z-pair directly, without using an item-by-item examination. It is based on

- the construction of a special form of representation for the input rational function $T(n, k)$,
- a direct construction of the minimal telescopers for each member of this representation,
- the use of Least Common Left Multiple (lclm) computation.

Telescopers of a Sum of Rational Functions

Lemma.

$$
\begin{array}{ccccccccc}
F(n,k) & = & F_1(n,k) & + & F_2(n,k) & + & \cdots & + & F_s(n,k) \\
 & & \Downarrow & & \Downarrow & & & & \Downarrow \\
 & & (L_1, G_1) & & (L_2, G_2) & & \cdots & & (L_s, G_s)
\end{array}
$$

$\mathrm{lclm}(L_1, \ldots, L_2)$ is a telescoper for $F = F_1 + \cdots + F_s$.

Example 5.

$$F = F_1 + F_2, \; F_1 = \frac{1}{5\,n - 4\,k - 4}, \; F_2 = \frac{1}{5\,n - 4\,k + 6}.$$

$$
\left.\begin{array}{ccccc}
F_1 & \stackrel{\mathcal{Z}}{\Longrightarrow} & L_1 & = & E_n^4 - 1 \\
F_2 & \stackrel{\mathcal{Z}}{\Longrightarrow} & L_2 & = & E_n^4 - 1
\end{array}\right\} \Longrightarrow \mathrm{lclm}(L_1, L_2) = E_n^4 - 1.
$$

$$F_1 + F_2 \quad \stackrel{\mathcal{Z}}{\Longrightarrow} \quad L = E_n^2 - 1.$$

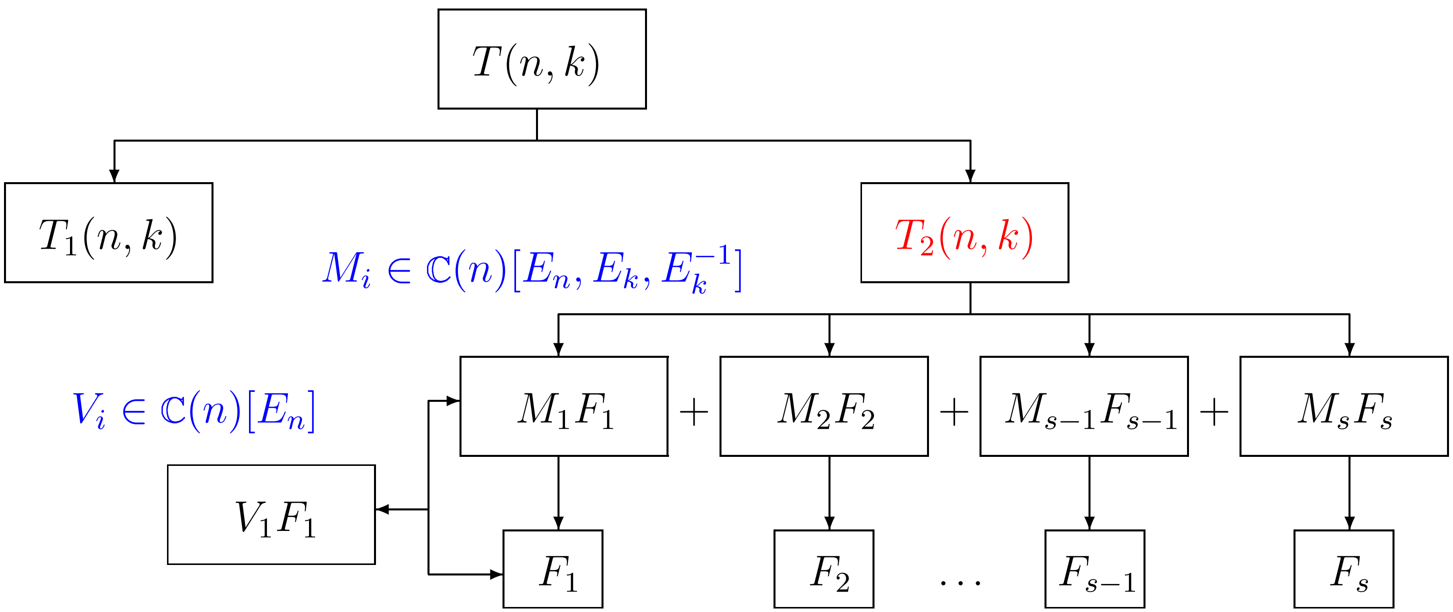
The Minimal Telescoper of a Sum

Lemma.

$$
\begin{array}{ccccccccc}
F(n,k) & = & F_1(n,k) & + & F_2(n,k) & + & \cdots & + & F_s(n,k) \\
 & & \Downarrow & & \Downarrow & & & & \Downarrow \\
 & & (L_1, G_1) & & (L_2, G_2) & & \cdots & & (L_s, G_s)
\end{array}
$$

For any telescoper L^* of F, if L^* is also a telescoper for each F_i, $i = 1, \ldots, s$, then $\mathrm{lclm}(L_1, \ldots, L_s)$ is the minimal telescoper for F.

The General Approach



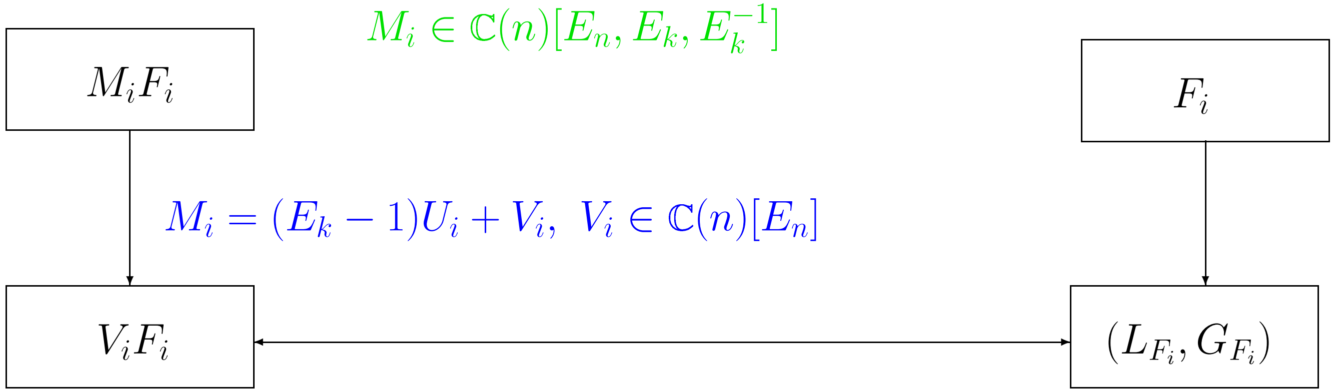
$$F_i = \frac{1}{(a_i\, n + b_i\, k + c_i)^{m_i}}$$

$$a_i,\, b_i \in \mathbb{Z},\ \ b_i > 0,\ \gcd(a_i, b_i) = 1,\ c_i \in \mathbb{C},\ m_i \in \mathbb{N} \setminus \{0\}$$

The Minimal Telescoper for $M_i\, F_i$

$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}},\ a_i,\, b_i \in \mathbb{Z},\, b_i > 0,\, \gcd(a_i, b_i) = 1,\, c_i \in \mathbb{C},\, m_i \in \mathbb{N} \setminus \{0\}$$

$$M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$$



$$L_i \in \mathcal{T}(M_i F_i) \iff L_i \in \mathcal{T}(V_i F_i)$$

$$(L_i, G_i) \in \mathcal{P}(V_i F_i) \Longrightarrow (L_i, G_i + L_i \circ U_i F_i) \in \mathcal{P}(M_i F_i)$$

$$\mathrm{lclm}(V_i, L_{F_i}) = L_1 \circ V_i = L_2 \circ L_{F_i},\ L_1,\ L_2 \in \mathbb{C}(n)[E_n]$$

$(L_1, L_2 G_{F_i})$ is the minimal Z-pair for $V_i F_i$

The Minimal Telescoper for F_i

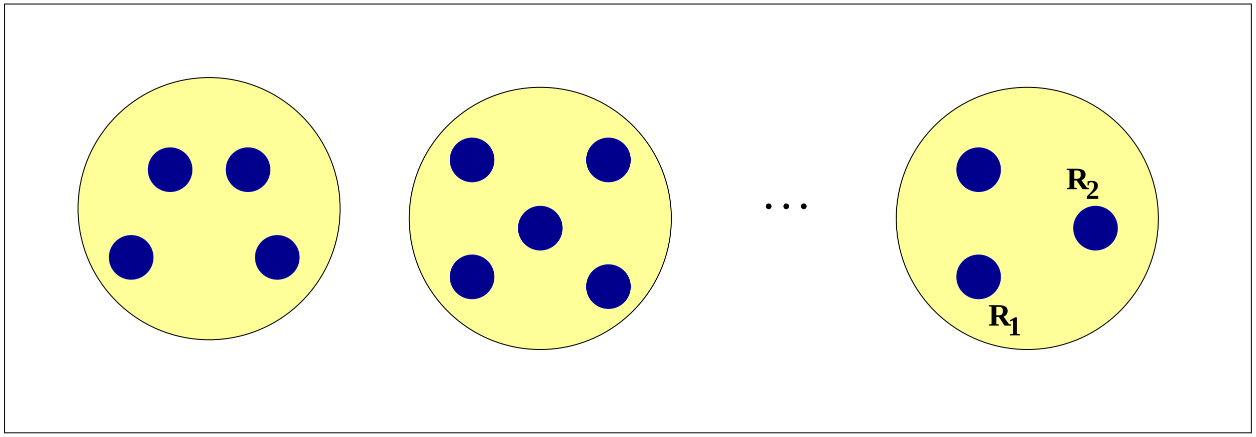
$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}},$$
$$a_i,\ b_i \in \mathbb{Z},\ b_i > 0,\ \gcd(a_i, b_i) = 1,\ c_i \in \mathbb{C},\ m_i \in \mathbb{N} \setminus \{0\}.$$

$\left(E_n^{b_i} - 1,\ W_i F_i\right)$ is the minimal Z-pair for F_i where

$$W_i = \begin{cases} \sum_{j=0}^{a_i - 1} E_k^j & a_i > 0 \\ \sum_{j=1}^{-a_i} -E_k^{-j} & \text{otherwise}. \end{cases}$$

A Special Form of Representation

The Equivalence Relation "$\sim$"



$$R_1 = \frac{r_1(n)}{(a_1 n + b_1 k + c_1)^{m_1}} \qquad a_i, b_i \in \mathbb{Z},\ b_i > 0,\ \gcd(a_i, b_i) = 1,$$

$$R_2 = \frac{r_2(n)}{(a_2 n + b_2 k + c_2)^{m_2}} \qquad c_i \in \mathbb{C},\ m_i \in \mathbb{N} \setminus \{0\},\ 1 \leq i \leq 2.$$

$$R_1 \sim R_2 \Longleftrightarrow (m_1 = m_2,\ a_1 = a_2,\ b_1 = b_2,\ c_1 - c_2 \in \mathbb{Z} \setminus \{0\}).$$

Experiment

$$T = \sum_i M_i F_i, \; M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}],$$

$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}}, \; a_i, \; b_i \in \mathbb{Z}, \; \gcd(a_i, b_i) = 1, \; b_i > 0, \; c_i \in \mathbb{C}, \; m_i \in \mathbb{N} \backslash \{0\}.$$

S_1 : $T = M_1 F_1 + M_2 F_2,$

$M_l = (r_{l_1}(n)/r_{l_2}(n)) E_n^i E_k^j, \; i \in \{1, 2, 3\}, \; j \in \{-3, \ldots, 3\},$

$\deg r_{l_s}(n) = 1, \; 1 \le s \le 2.$

S_2 : $T = M_1 F_1, M_1 = (r_1(n)/r_2(n)) E_n^i E_k^{-j} + (r_3(n)/r_4(n)) E_n^u E_k^v,$

$i, j, u, v \in \mathbb{N}, \; i + j = u + v = 2,$

$\deg r_1(n) = \deg r_3(n) = 1, \; \deg r_2(n) = \deg r_4(n) = 2.$

S_3 : $T = M_1 F_1 + M_2 F_2 + M_3 F_3.$

We ran the tests using an implementation of the direct algorithm and the original form of Zeilberger's algorithm.

Note. We enforced a limit of 2,000 seconds on each input rational function in the tests.

Table 1: Total time and space requirements for $\mathcal{D}$ and $\mathcal{Z}$.

	Completed		Timing (seconds)		Memory (kilobytes)	
	$\mathcal{D}$	$\mathcal{Z}$	$\mathcal{D}$	$\mathcal{Z}$	$\mathcal{D}$	$\mathcal{Z}$
S_1	20	15	12.15	3127.84	54,159	8,095,930
S_2	20	18	12.43	2635.94	54,653	7,873,146
S_3	20	0	959.07	–	3,864,026	–

Note. All the reported timings were obtained on a 400Mhz SUN SPARC SOLARIS with 1Gb RAM.

Hypergeometric Case: a Lower Bound

S.A. Abramov, H.Q. Le. A Lower Bound for the Order of the Telescopers for a Hypergeometric Term. In O. Foda, Ed., *Formal Power Series and Algebraic Combinatorics*, on CD, 2002.

The approach is to define operators $M \in \mathbb{C}[n, E_n]$, called crushing operators, and show that if $L \in \mathbb{C}[n, E_n]$ is a telescoper for the input hypergeometric term $T(n, k)$, then L is also a crushing operator for $T(n, k)$. The problem is then reduced to computing a lower bound for the order of the minimal crushing operator for T.

Rational Normal Forms

S.A. Abramov, M. Petkovšek. Canonical representations of hypergeometric terms. *Proc. FPSAC 2001*, 1–10.

Definition. Let $R \in \mathbb{K}(k) \setminus \{0\}$. If there exist

$$f_1, f_2, v_1, v_2 \in \mathbb{K}[k] \setminus \{0\} \text{ such that}$$

(i) $R = F \cdot \dfrac{E_k V}{V}$ where $F = \dfrac{f_1}{f_2}$, $V = \dfrac{v_1}{v_2}$, and $\gcd(v_1, v_2) = 1$,

(ii) $\forall\, h \in \mathbb{Z},\ \gcd(f_1, E_k^h f_2) = 1$ (F is *shift-reduced* w.r.t. k),

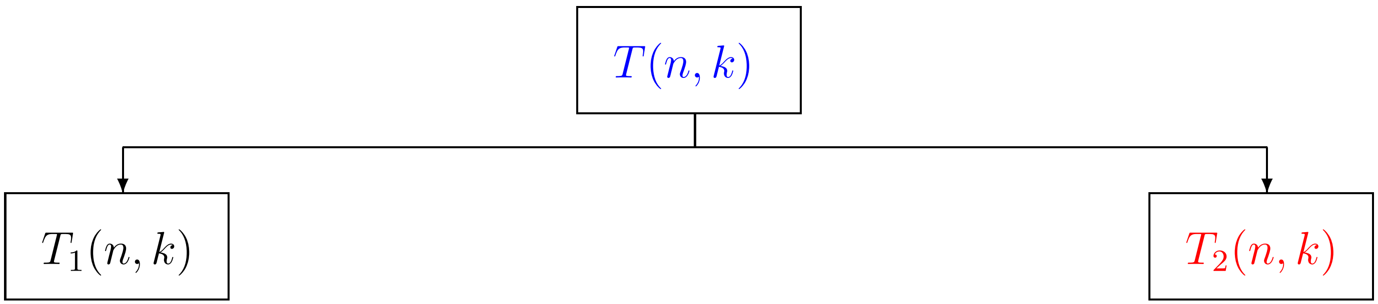
then

$$F \cdot \frac{E_k\, V}{V} \equiv (z,\ f_1,\ f_2,\ v_1,\ v_2) \text{ where } z = \mathrm{lcoeff}(F)$$

is a rational normal form (RNF) of R.

ADP: Hypergeometric Case

S.A. Abramov, M. Petkovšek. Rational Normal Forms and Minimal Decompositions of Hypergeometric Terms. *Journal of Symbolic Computation* **33**, No. 5, 521–543, 2002.



$$T(n,k) = (E_k - 1)T_1(n,k) + T_2(n,k)$$

$$(z, f_1, f_2, v_1, v_2) := \mathrm{RNF}_k(\mathcal{C}_k(T_2(n,k)))$$

Properties: $\forall p : p \,|\, v_2$

$$H_1 : E_k^h p \,|\, v_2 \Longrightarrow h = 0 \ (v_2 \text{ is shift-free w.r.t. } k)$$

$$H_2 : E_k^h p \,|\, f_1 \Longrightarrow h < 0, \ E_k^h p \,|\, f_2 \Longrightarrow h > 0$$

A Structure Theorem of Hypergeometric Terms

S.A. Abramov, M. Petkovšek. Proof of a Conjecture of Wilf and Zeilberger. *University of Ljubljana, Preprint series* **39**, 2001.

$$T(n,k)\left\{\begin{array}{rcl}\mathrm{RNF}_k(\mathcal{C}_k(T(n,k))) & = & F\cdot\dfrac{E_k\,V}{V}\\ & & \quad\downarrow\\ \mathcal{C}_n(T(n,k)) & = & D\cdot\dfrac{E_n\,V}{V}\end{array}\right.$$

for $D \in \mathbb{C}(n,k)$.

$$F,\ D \in Z_{n,k}$$

Crushing Operators

$T_2(n,k)$: non-summable part of the given hypergeometric term $T(n,k)$.

Consider $M \in \mathbb{C}[n, E_n] : MT_2 \neq 0$. Let

$$\mathrm{RNF}_k(\mathcal{C}_k(MT_2)) = F' \cdot \frac{E_k V'}{V'}, \; V' = \frac{v_1'}{v_2'}.$$

Then M is a crushing operator for T_2 (denoted by $CO(T_2)$) if each of the irreducible factors of v_2' does not have at least one of the two Properties H_1, H_2.

Proposition. If L is a telescoper for T_2, then L is a crushing operator for T_2.

A Lower Bound for $MCO(T_2)$

Theorem.

$$T_2(n,k)\left\{\begin{array}{rcl} \mathrm{RNF}_k(\mathcal{C}_k(T_2(n,k))) & = & F\cdot\dfrac{E_k\,V}{V},\ V=\dfrac{v_1}{v_2} \\ \\ \mathcal{C}_n(T_2(n,k)) & = & D\cdot\dfrac{E_n\,V}{V},\ D=\dfrac{d_1}{d_2} \end{array}\right.$$

Let $\operatorname{ord} MCO(T_2)=\rho$. Let p be an integer-linear factor of v_2, $\deg_k p = 1$. Then

(i) $\exists\, h\in\mathbb{Z}:$

$$E_k^h\,p \,|\, E_n v_2\cdot E_n^2 v_2\cdots E_n^{\rho} v_2\cdot d_2\cdot E_n d_2\cdots E_n^{\rho-1} d_2; \qquad (3)$$

(ii) Let ρ_p be the minimal value of ρ such that (3) is satisfied. Then $\operatorname{ord} MCO(T_2)\geq \max_{p\,|\,v_2}\ \rho_p$.

Algorithm Description

For each integer-linear factor p of v_2, $\deg_k p = 1$, compute the minimal value of ρ in the pair (ρ, h), $h \in \mathbb{Z}$, $\rho \in \mathbb{N} \setminus \{0\}$ such that

(i) $E_k^h p \,|\, E_n v_2 \cdot E_n^2 v_2 \cdots E_n^\rho v_2$, or

(ii) $E_k^h p \,|\, d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$.

Note. By the "structure theorem", $d_2 \in Z_{n,k}$.

Example

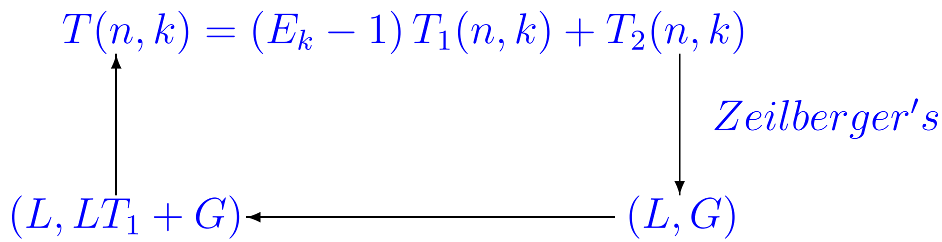
Consider the class of hypergeometric terms of the form

$$T(n,k) = \frac{1}{(a_1 n + b_1 k + c_1)\,(a_2 n + b_2 k + c_2)!},$$

$a_1,\ b_1,\ a_2,\ b_2 \in \mathbb{Z},\ \gcd(a_1, b_1) = 1,\ b_1 \neq 0,\ a_1 \neq a_2$ or $b_1 \neq b_2$.

One can prove that the computed lower bound for the orders of the telescopers for $T(n,k)$ is $|b_1|$.

Experiment



For $b \in \mathbb{N} \setminus \{0\}$, $j \in \{1, 3\}$, let

$$T_1(n,k) = \frac{1}{(nk-1)(n - b\,k - 2)^j (2\,n + k + 3)!},$$

$$T_2(n,k) = \frac{1}{(n - b\,k - 2)(2\,n + k + 3)!}.$$

We are interested in computing the minimal telescopers for

$$T(n,k) = (E_k - 1)\, T_1(n,k) + T_2(n,k).$$

Table 2: Time and space requirements for $\mathcal{L}$ and $\mathcal{Z}$.

		Timing (seconds)		Memory (kilobytes)	
j	b	$\mathcal{L}$	$\mathcal{Z}$	$\mathcal{L}$	$\mathcal{Z}$
1	1	6.49	5.35	27,838	24,702
	2	8.34	34.64	33,066	142,889
	3	11.13	124.53	44,233	535,736
	4	14.46	570.02	56,410	1,882,730
	5	25.79	2999.22	97,506	6,536,309
3	1	14.64	16.40	62,566	73,830
	2	17.24	228.59	68,304	770,529
	3	20.15	1286.51	78,701	3,074,051
	4	24.08	8771.08	91,844	10,766,646
	5	38.60	77663.68	139,823	33,423,168

Telescopers Package

S.A. Abramov, K.O. Geddes, H.Q. Le. Computer Algebra Library for the Construction of the Minimal Telescopers. *International Congress of Mathematical Software*, World Scientific, 319–329, 2002.