Les empilements de segments et la gravitation quantique.

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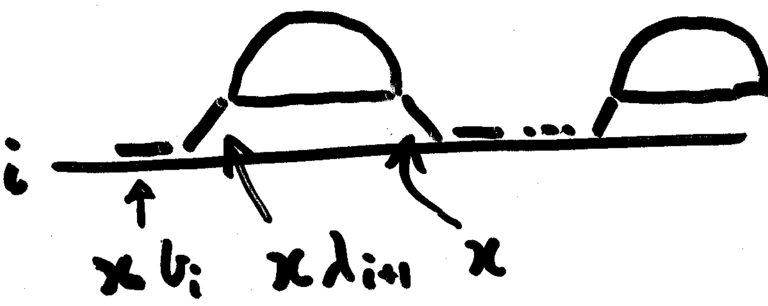
[Travail avec X. Viennot (LaBRI)]

I Fractions continues

"J-fraction" (Jacobi)

D_i = development en fraction continue

$$= \frac{1}{1 - b_i x - \lambda_{i+1} x^2 D_{i+1}}$$



Chemin de Motzkin.

$$D_0 = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{1 - \ddots}}}$$

Le convergent:

$$\frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

$D_{i,k}$ = Hauteur bornée à k.

$\therefore D_{k+1} = 0.$

où $P^*_n = x^n P_n(\frac{1}{x})$ (polynômes)

2| Polynômes Orthogonaux. P_n, P_n^*

$$P_n^*(x) = x^n P_n\left(\frac{1}{x}\right) = \det\left[I_n - x T_n\right] \quad \text{où}$$

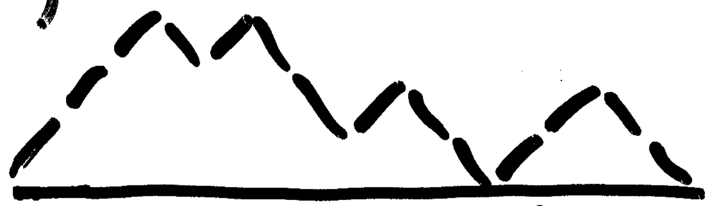
Récurrence à 3 termes

$$T_n = \begin{pmatrix} b_0 & 1 & & & 0 \\ \lambda_1 & b_1 & 1 & & \\ & \lambda_2 & b_2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \lambda_k & b_k \end{pmatrix}$$

$$P_{n+1}(x) = (x - b_n) P_n(x) - \lambda_n P_{n-1}(x)$$

<u>Cas Special</u> $(b_i = 0 \quad \lambda_i = 1)$

Chemins de Dyck



$$P_{n+1}^*(x) = P_n^*(x) - x^2 P_{n-1}^*(x) \qquad D'_{0,n} = \frac{P_n^{*\prime}(x)}{P_{n+1}^{*\prime}(x)}$$

↳ Récurrence des polynômes Fibonacci !

3| Chemins de Dyck

$$F_t(x,y) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \ddots \cfrac{}{1 - \cfrac{x^2}{1 - x^2 y}}}}} \qquad F_t = \frac{1}{1 - x^2 F_{t-1}} \qquad F_0 = y$$

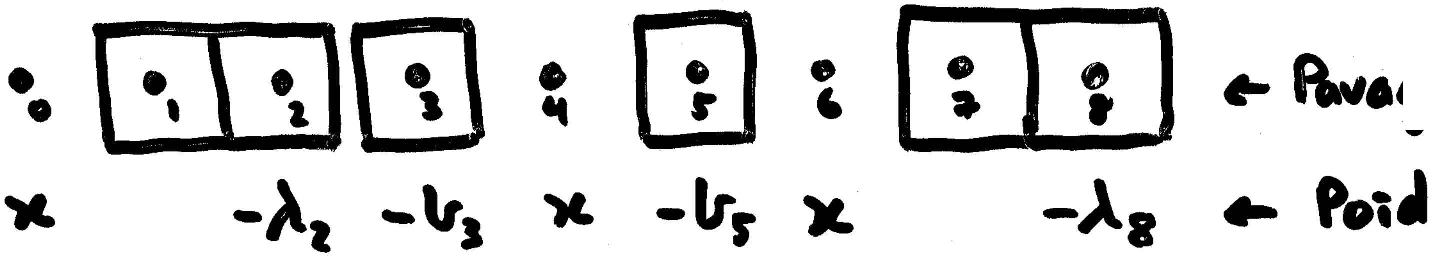
λ_i constant
↳ homogeneous

Alors $F_t = \dfrac{A_t y + B_t}{C_t y + D_t}$ transformation linéaire fractionale.

$A_t, \dots D_t$ sont de la même famille, satisfaisant la récurrence

$$\phi_{t+1} = \phi_t - x^2 \phi_{t-1}$$

4] $P_n(x)$ – Pavage du segment $[0, n-1]$



$b_i = 0, \lambda_i = 1$:

$P_{n+1}(x) = x\, P_n(x) - P_{n-1}(x)$ polynômes Chebyshev.

$P^*_{n+1}(x) = F_{n+1}(x^2)$, F_n - Fibonacci.

Mais: $2\cos(\theta)\sin(n\theta) = \sin(n+1)\theta + \sin(n-1)\theta$

alors, soit $x = 2\cos(\theta)$

$$\therefore A(\theta)\sin(n+1)\theta = x\, A(\theta)\sin(n\theta) - A(\theta)\sin(n-1)\theta$$

$$\Rightarrow P_n(2\cos\theta) = \sin((n+1)\theta)/\sin\theta .$$

5] Expression trigonometrique pour $P_n(x)$, D_{kn}.

$P_n(x_j) = 0 \Rightarrow \sin(n+1)\theta_j = 0 \Rightarrow (n+1)\theta_j = j\pi$, $j \in \mathbb{N}$

$\therefore \theta_{jn} = \frac{j\pi}{n+1}$, $j \in [1, n]$.

Alors, racines pour $F_n(4\cos^2\theta)$ sont θ_{jn}, $j \in [1, \lfloor \frac{n}{2} \rfloor]$

de Bruijn, Knuth, Rice

$$\hookrightarrow D_{0n} = \frac{xF_n(x)}{F_{n+1}(x)} = \sum_{1 \le k \le \frac{n}{2}} \frac{\tan^2\theta_{kn}}{(n+1)(1 - 4\cos^2\theta_{kn}\, x)} + a_n + b_n x$$

] N.B. L'equation fonct^{nelle} de ζ de Riemann.
Ref: Exposé de Phillipe Biane 1/12/03. (donné par (1) + (2)!)

(1) Principe de reflexion

$$Pr\left(\begin{smallmatrix}\text{hauteur}\\ \text{max.}\end{smallmatrix} \leq m\right) = 2^{-(2n-1)} \sum_k \binom{2n-2}{n-1-k(m+1)} \binom{2n-2}{n-2-(k+1)(m+1)}$$

$$(m = \sqrt{\pi n}\, y) \quad \sim \sum_k (\sim) \exp(-\pi k^2 y^2)$$

(2) En diagonalisant le matrice de transfer

$$Pr(\quad) = \frac{2}{m} \sum_{k=1}^{m} \sin^2\left(\frac{k\pi}{m+1}\right) \cos^{2n-2}\left(\frac{k\pi}{m+1}\right) \quad \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & \ddots & \end{pmatrix}$$

$$\sim \sum_k (\sim) \exp(-\pi k^2 / y^2)$$

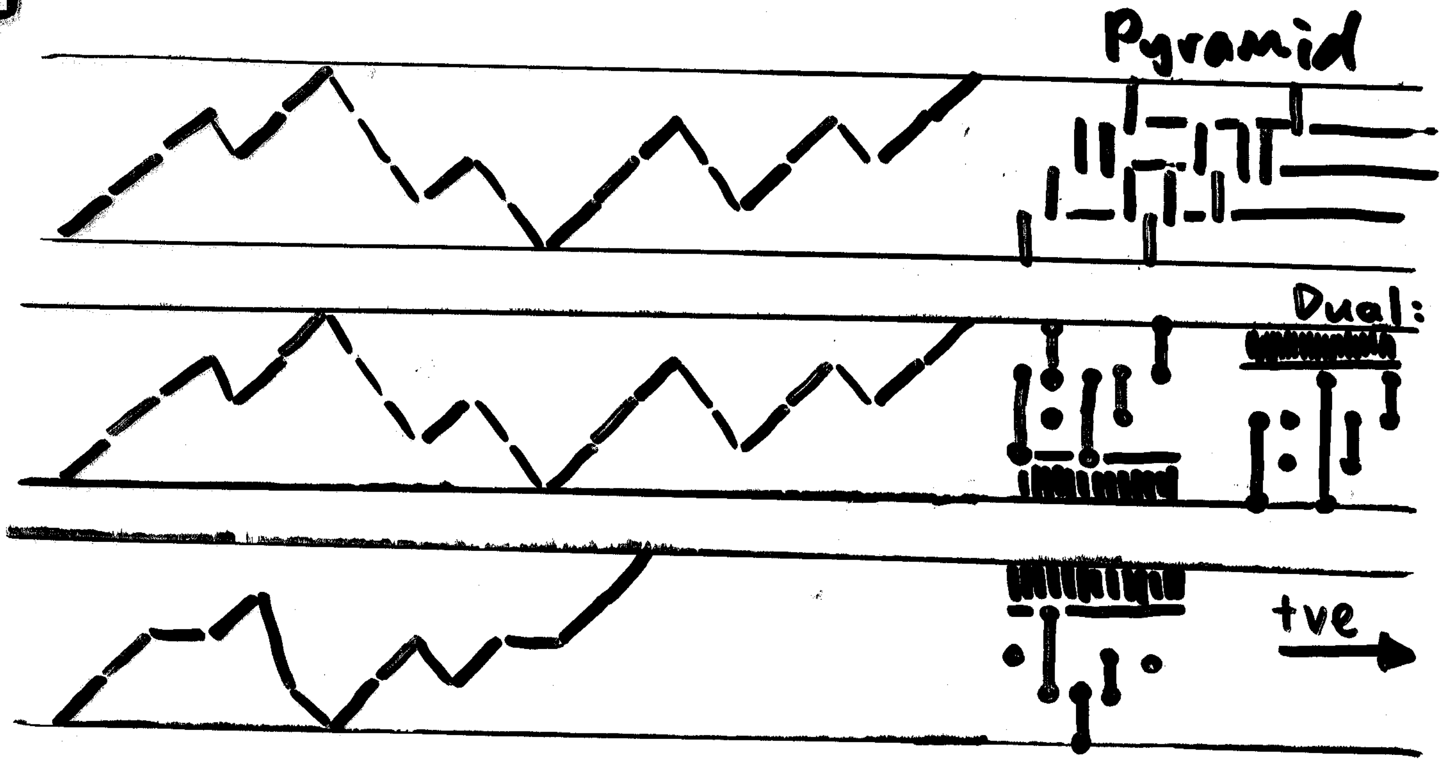
] Empilements des pavages

Rappelons $D_{0n} = \dfrac{\delta P^*_n(x)}{P^*_{n+1}(x)}$ $(\lambda_i = 1, b_i = 0)$

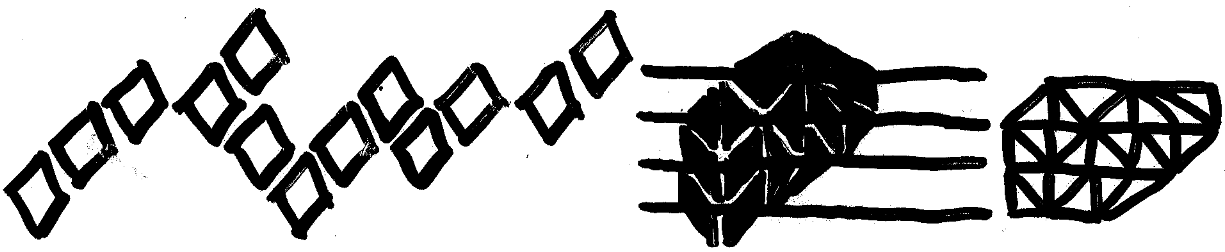
Alors, il y a un lien entre:

Chemins de Dyck
$\updownarrow$
Fractions Continues
$\updownarrow$
Polynômes Orthogonaux

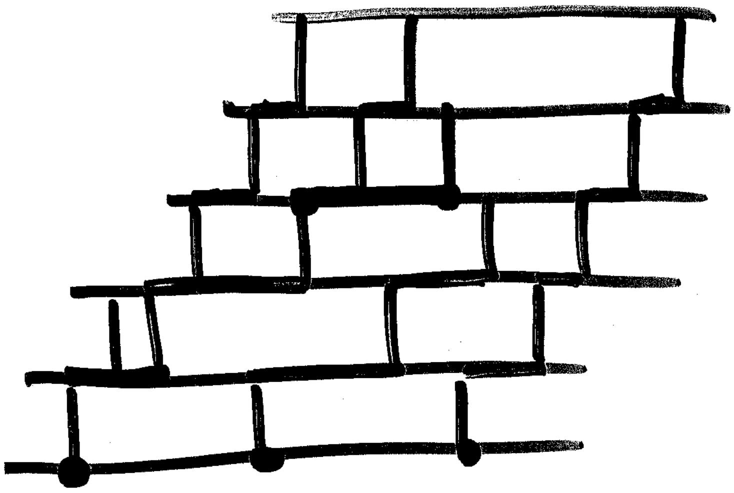
8

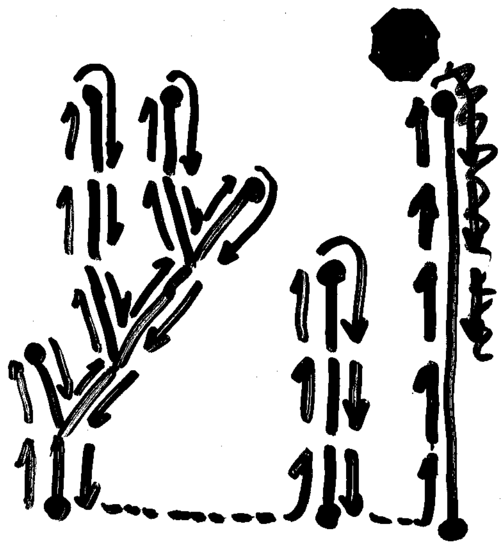


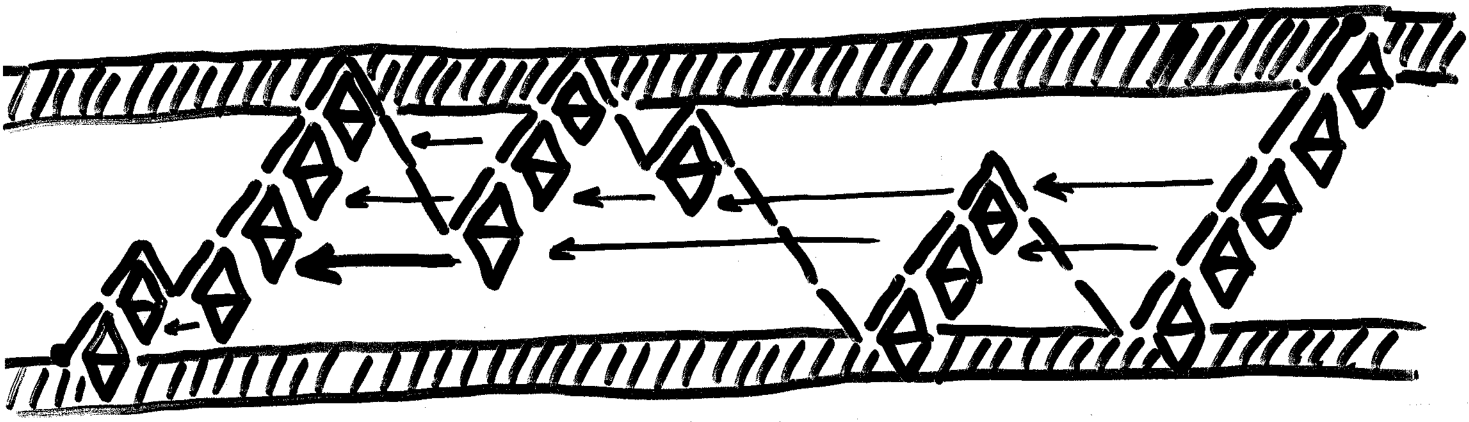
8a



I La bijection



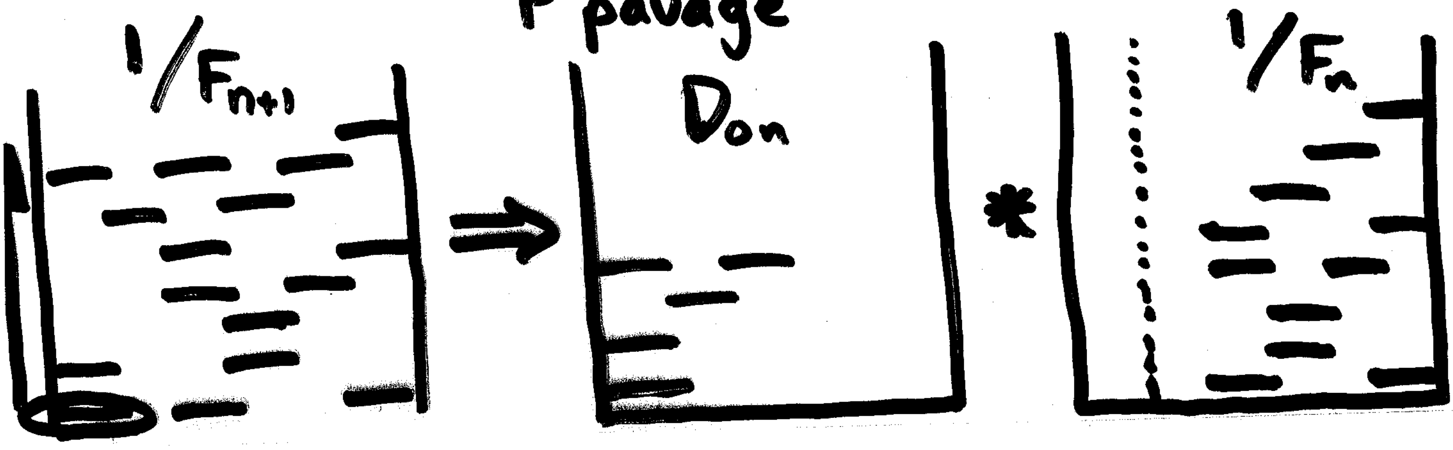




b) Le lemme d'Inversion

Soit $F(x)^{+1} = \sum_{T \text{ empilement}} (+1)^{|T|} v(T)$

Alors $F(x)^{-1} = \sum_{P \text{ pavage}} (-1)^{|P|} v(P)$

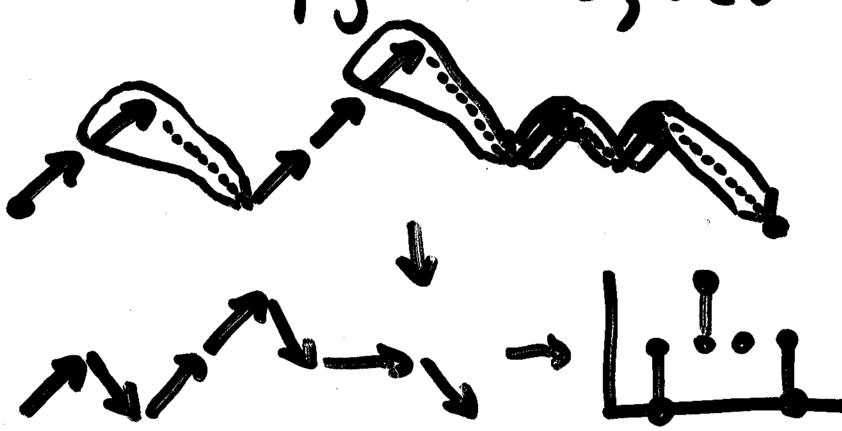


1] Les Chemins de Lukasiewicz

Maintenant qu'on a une équivalence entre chemins de Dyck et demi-pyramids, des segments ????

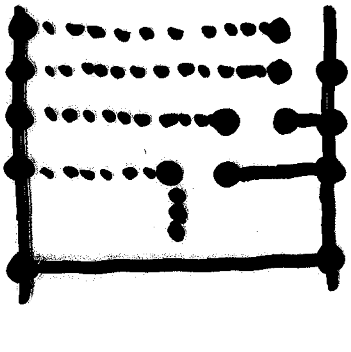
Chemin de : Dyck

Lukasiewicz



2] Demi-pyramids de segments

$\dfrac{S_n(x)}{S_{n+1}(x)}$ où $S_n(x)$ = empilement trivial (i.e. pavage) de segments sur $[0,n]$.

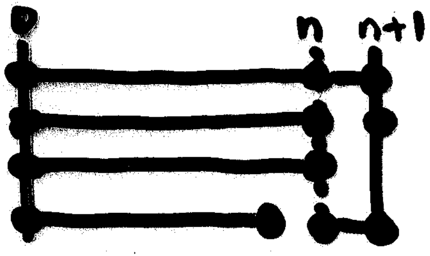


$$= \sum_{P \text{ pavage}} x^{\text{longeur}} (-t)^{\#\text{segm}}$$

$$t \; (-1)^{|P|} \, v(P)$$

$$S_{n+1} = S_n - t \sum_{i=0}^{n-1} S_i \, x^{n-i} - t x^{n+1}$$

Mais ! — on attend des fonctions (polynômes) orthogonaux.

3] 

$$S_{n+1} = S_n(1-t) + x\cancel{t}(S_n - S_{n-1})$$

dernier segment alongé

n.b. $S_n - S_{n-1}$ = pavage qui touch la borne droite.

$$\therefore\ S_{n+1}\left(\overset{t}{\underset{\lambda}{x}}\right) = (1+x-t)\,S_n(x,t) - x\,S_{n-1}(x,t)$$

$$= (1+x-t)^n F_n\left(\frac{x}{(1+x-t)^2}\right)$$

$$= (1+x-t)^n \frac{1-\lambda^{n+1}}{(1-\lambda)(1+\lambda)^n} \quad \text{où } \frac{\lambda}{(1+\lambda)^2} = \frac{x}{(1+x-t}$$

4] Cas $t=x$. Les demi-pyramids de dominos/seg

Soit $Q(x)$ la fonction génératrice des demi-pyra^s de domino

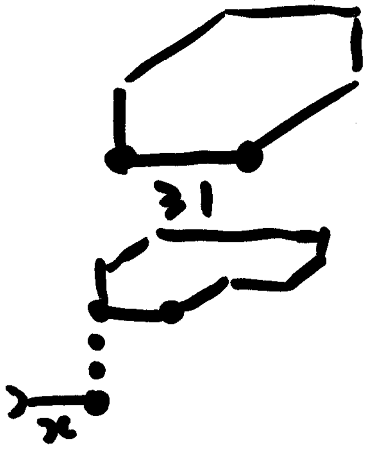
$$Q(x) = \sum_{n\geq 0} \frac{x^n}{F_n F_{n+1}} \left[= \sum_{n\geq 0}\left(\frac{F_n}{F_{n+1}} - \frac{F_{n-1}}{F_n}\right)\right]$$

alors $x = \frac{Q}{(1+Q)^2}$

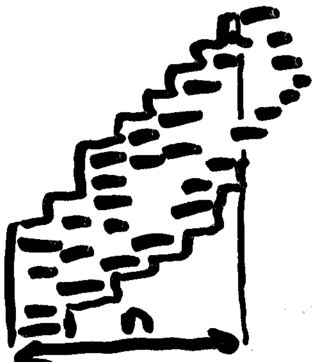
$$Q(x,t) = \sum_{n\geq 0} \frac{x^n}{S_n S_{n+1}} \quad \text{qui donne} \quad \frac{\lambda}{(1+\lambda)^2} = \frac{x}{(1+x-t)^2}$$

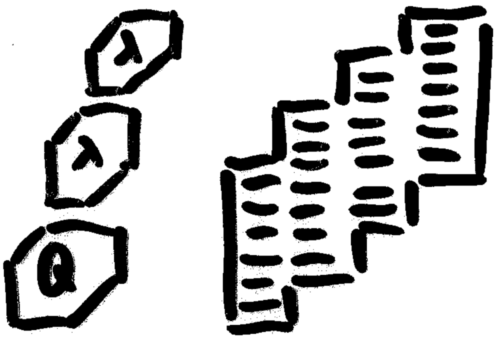
$$\left(= \frac{1+\lambda}{1+x-t}\right) \qquad \lambda\Big|_{t=x} = Q-1.$$

5 λ – (valeur propre de la matrice de transfer)

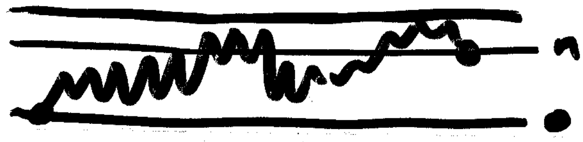
$1+\lambda = (1-t)\, Q$ 

$+\, x\, Q$





λ^n crée une condition de born
du haut escalière.

$\sim$ 

Remarquons:

$S'_{a+1,b} = (1+x-y)\, S'_{a,b} - x\, S'_{a-1,b}$

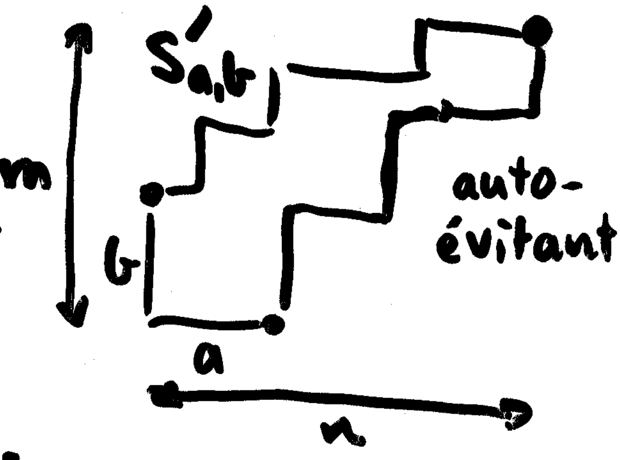
$\Rightarrow S'_{a,b} = u^a v^b$

où $x = u(1-v)$, $y = v(1-u)$.

$\Rightarrow u = x + SP$, $v = y + SP$

Si $t=y$, on a

$Q(x,t) = 1 + \frac{SP}{x}$ $\quad SP = uv$



$S'_{a,b} = \sum s'_{a,b}\, x^n y^m$

3) Empilements connexes de segments.

Ref : Bousquet-Mélou, Rechnitzer

$$C(x,x,u) = \frac{1}{u} - 1 - \left[\sum_{n\geq 0} \frac{u^{n+1}}{F_n}\right] \qquad \mu = 4.58789\ldots$$

$$C(x,x,1) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k+1}(1+Q)}\right]} \sim A\mu^n$$

Remplaçant F_n par S_n,

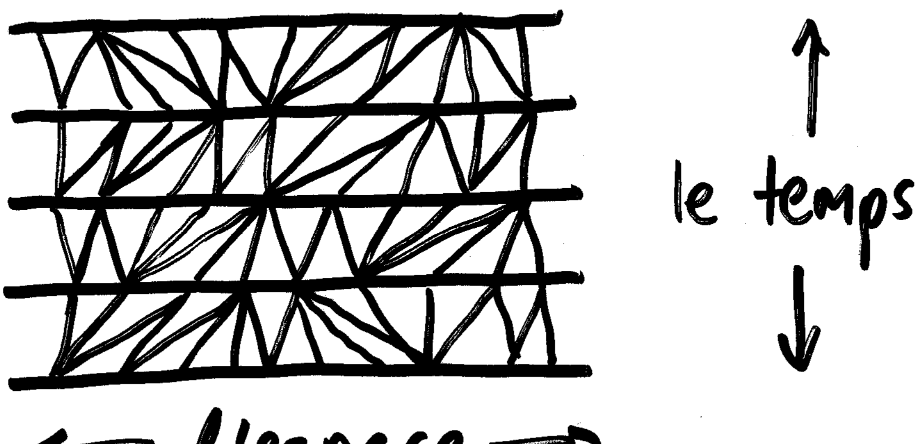
$$c(x,x,1,t) = \frac{Q}{(\lambda^2-1) \sum_{n\geq 0} \frac{\lambda^n}{1-Q\lambda^n}}$$

I: Le contexte.

Ref: "A discrete history of the Lorenzian path integral." R. Loll. ArXiv: hep-th/0212340 (13/1/03)

Def$^{\underline{n}}$: La gravitation quantique.

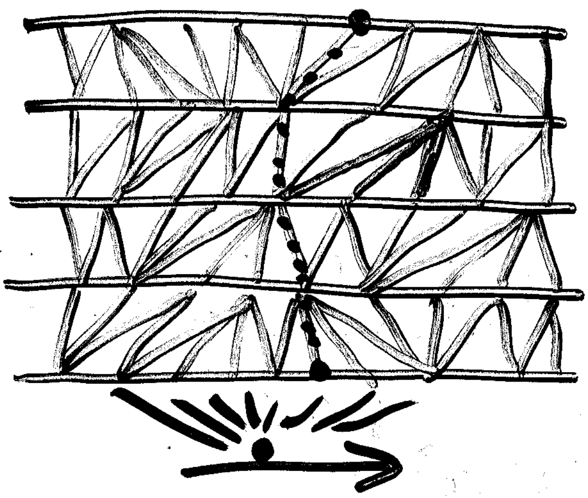
Une description quantique de la géométrie de l'espace-temps qui est consistante avec la physique aux petites et grandes échelles.



L'approche : On regarde le niveau régularisé par une triangulation discrète avant de regarder la limite continue.

La gravitation quantique <u>Lorenzienne</u>.

L'univers
de
dimension
(1+1)

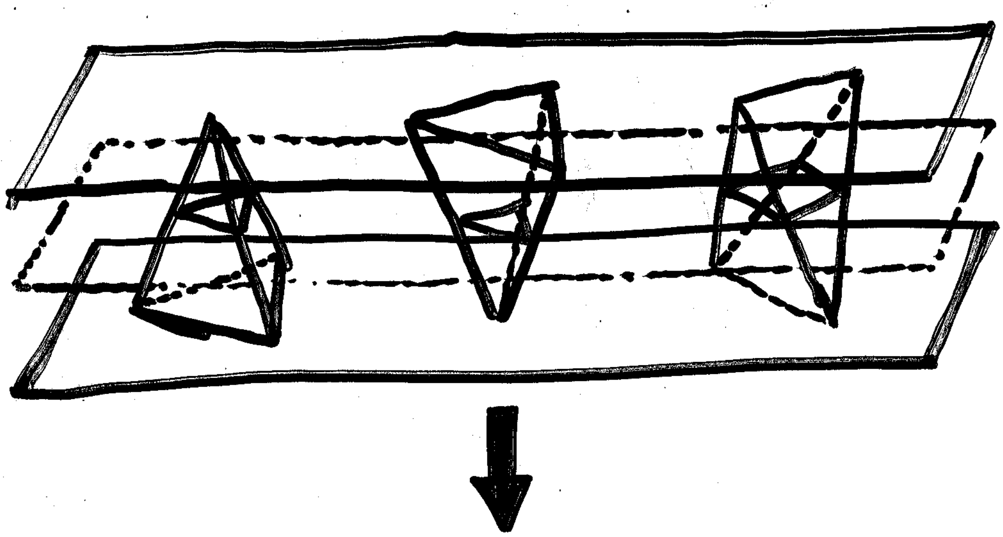


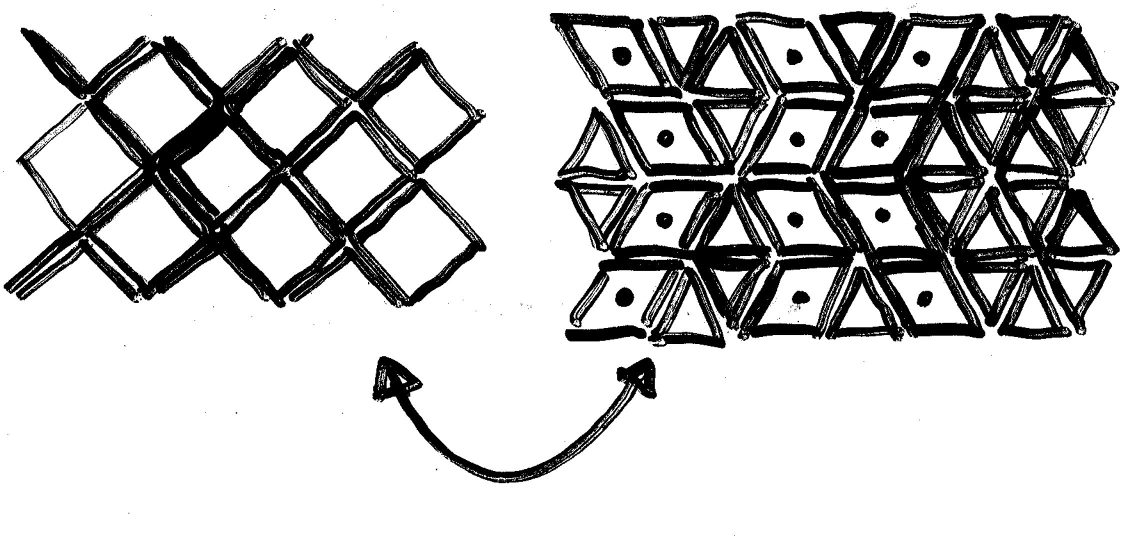
temps

l'espace

Ref:
Ambjørn,
Loll (1998)

L'univers
de
dimension
(2+1)

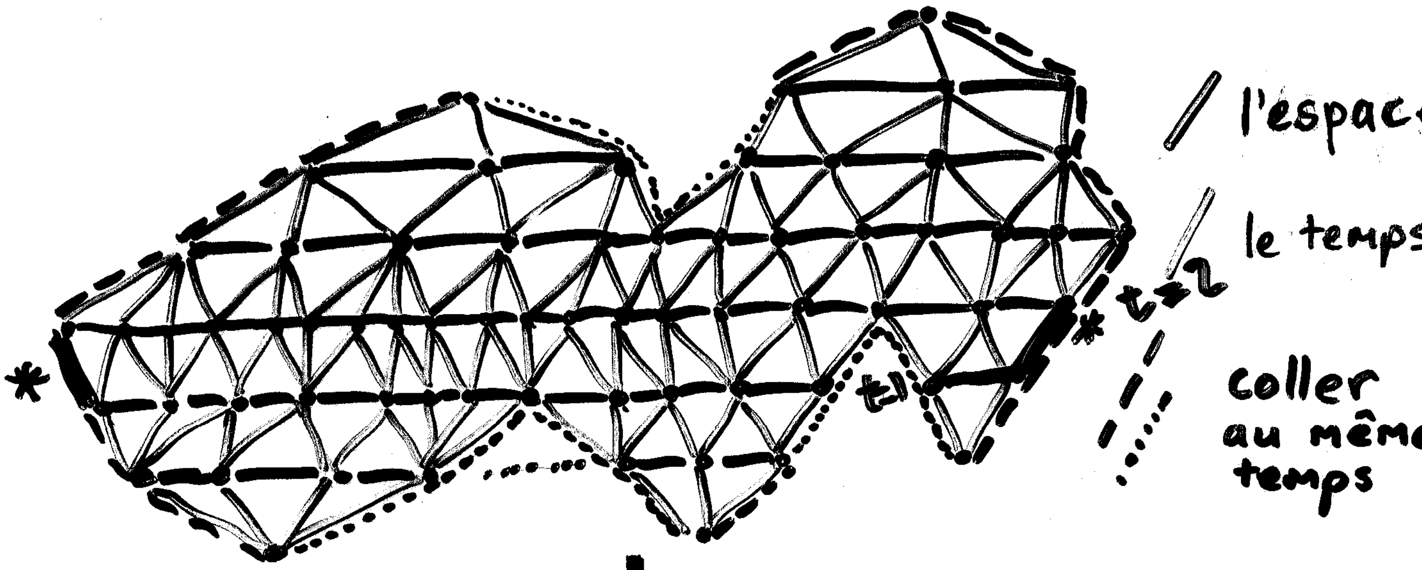




Ref:
Dittrich,

La gravitation quantique.

On représente l'espace-temps comme une triangulation :



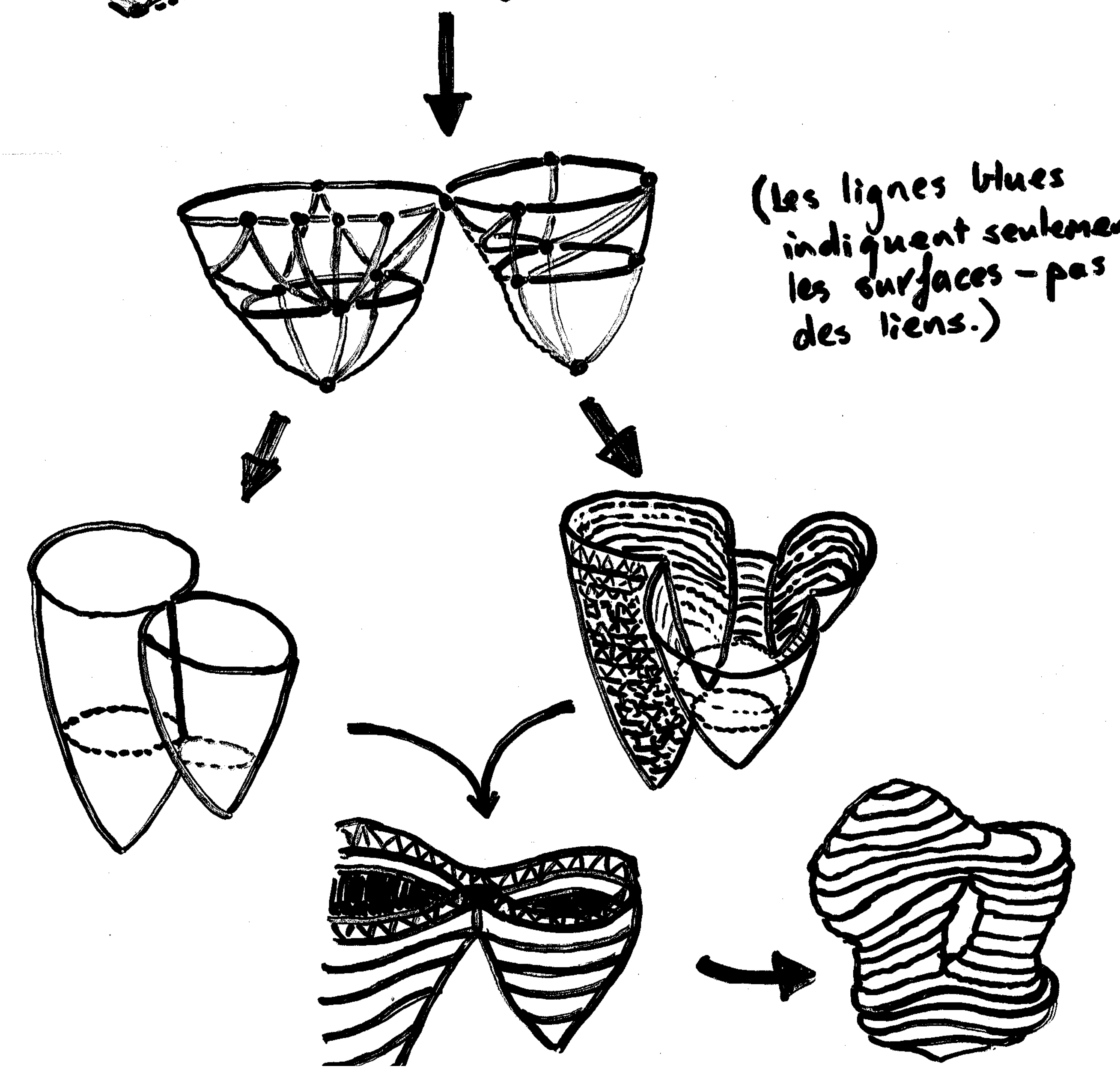


Fig 2

p 2a

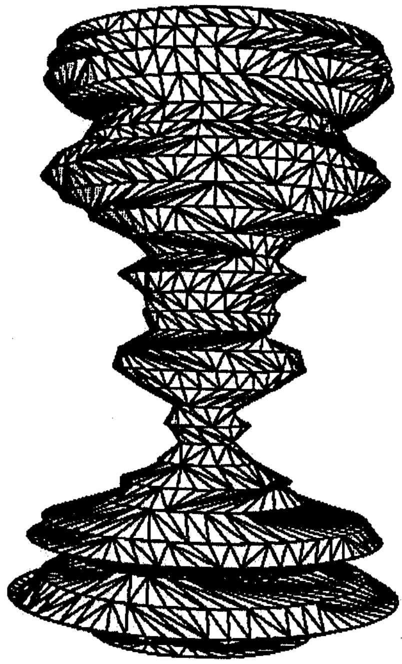


Figure 1: A typical discrete history of pure Lorentzian gravity with volume N=1024.

p 2b

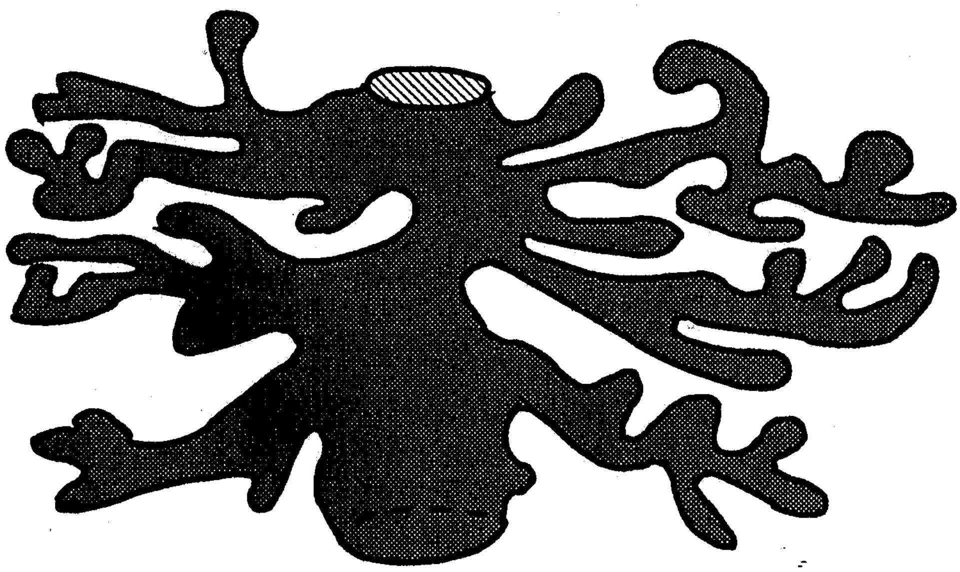


Figure 4. The fractal baby-universe structure of 2d Euclidean gravity, artist's impression.

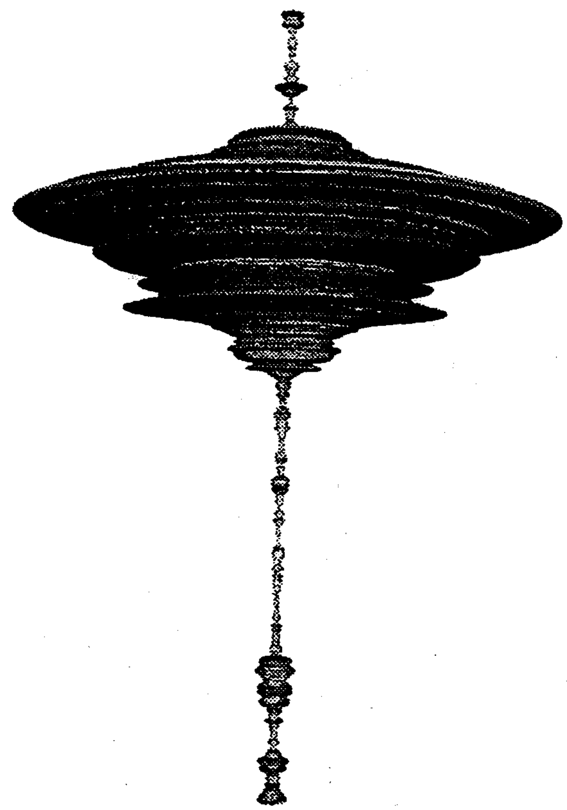


Figure 6. A typical Lorentzian geometry in the presence of eight Ising models, at volume $N_2 = 73926$ and total proper time $t = 333$.

p2d

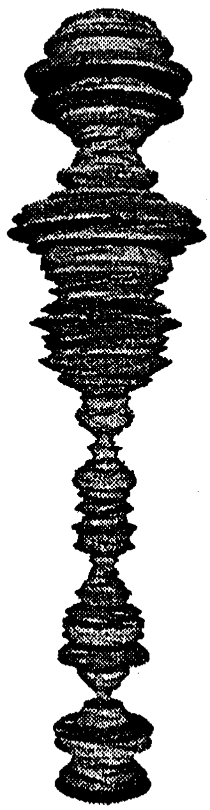
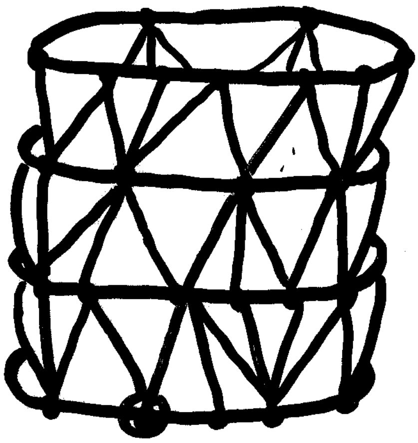


Figure 5. A typical Lorentzian geometry in the presence of one Ising model, at volume $N_2 = 18816$ and total proper time $t = 168$.

I : Les modèles.

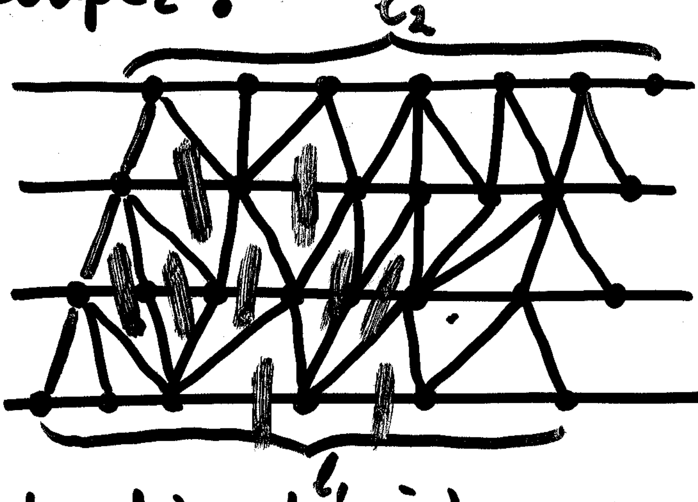
- 2-D (1-espace, 1-temps)
- cylindrique
  ↳ l'espace est un boucle
  (voir Fig 1)
- la causalité
  ↳ une origine unique
  les 'univers-bébé' sont interdits.
  (voir Fig 2)

Ambjørn, Loll (1998) hep-th/9805108 2c



- Choisir un sommet du départ
- Prendre le lien à droite
- Coupez !

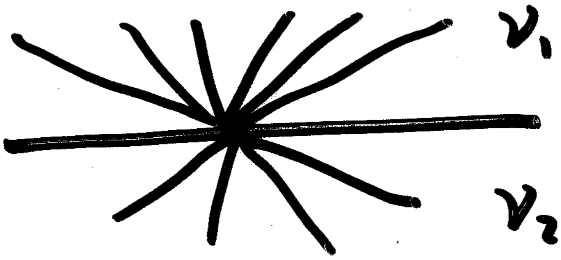
$\Rightarrow$



Les triangles à gauche doivent pointer vers le haut.

I Les modèles (cont.)

Le prochain paramètre qui nous intéresse est la courbure.



ν_1 liens du temps

ν_2 liens du temp

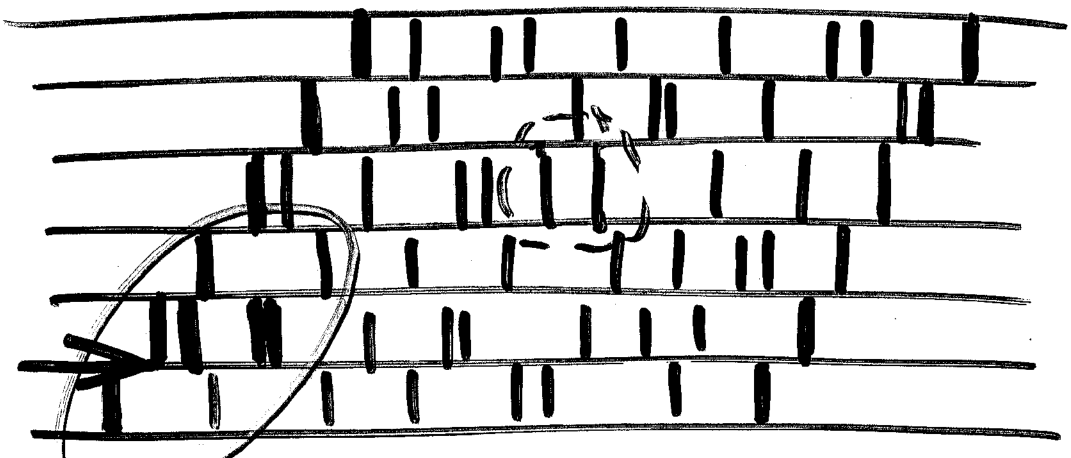
$$a^{(\nu_1 - 2 + \nu_2 - 2)} = a^{(\#\text{ liens } \underline{\text{total}} - 6)}$$

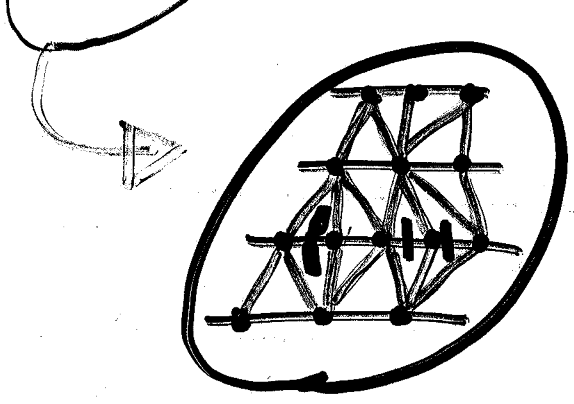
rouges inclus!

Di Francesco, Guitter, Kristjansen (1999) hep-th/9907084

Comment mettre la courbure ?

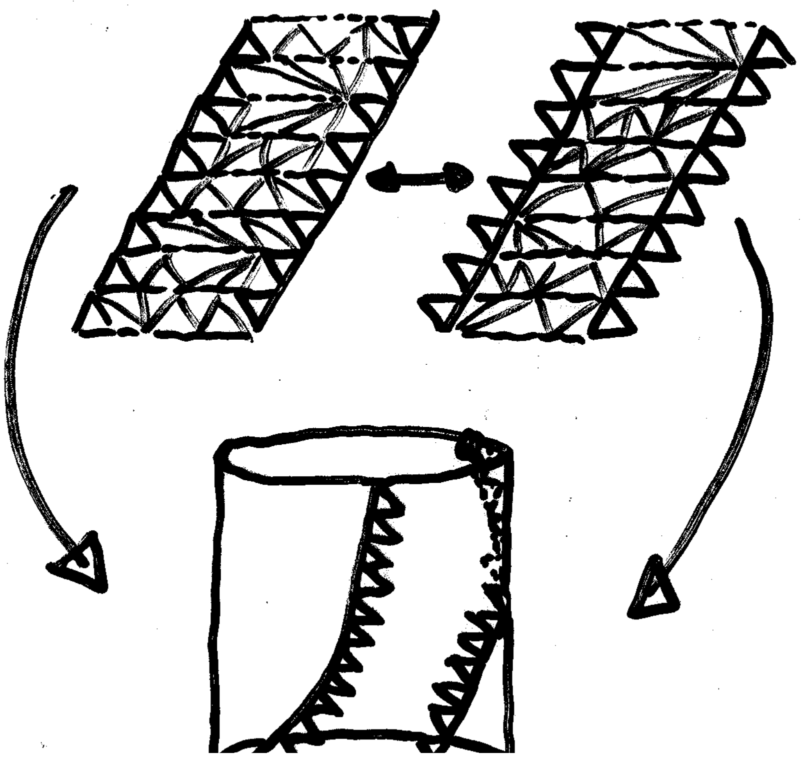
P. Di Francesco, E. Guitter, C. Kristjansen;
"Integrable 2D Lorenzian Gravity and Random Walks". Nucl. Phys. B 567 (2000) pp 515–553.
arXiv: hep-th/9907084





domino = 2 triangles
= g^2

valuation d'un domino de la borne = g

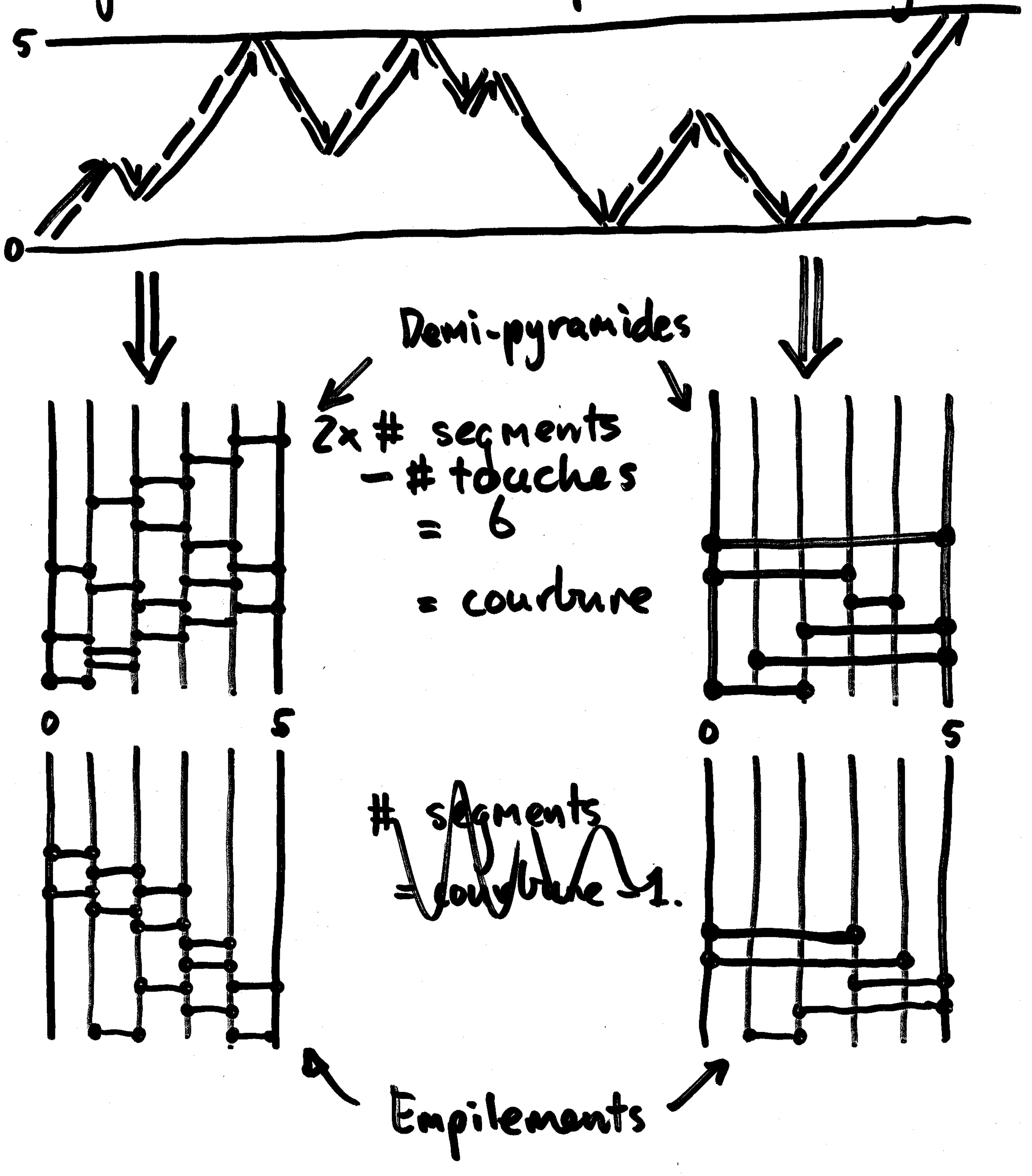


$G^{(p)}$

2 contours.

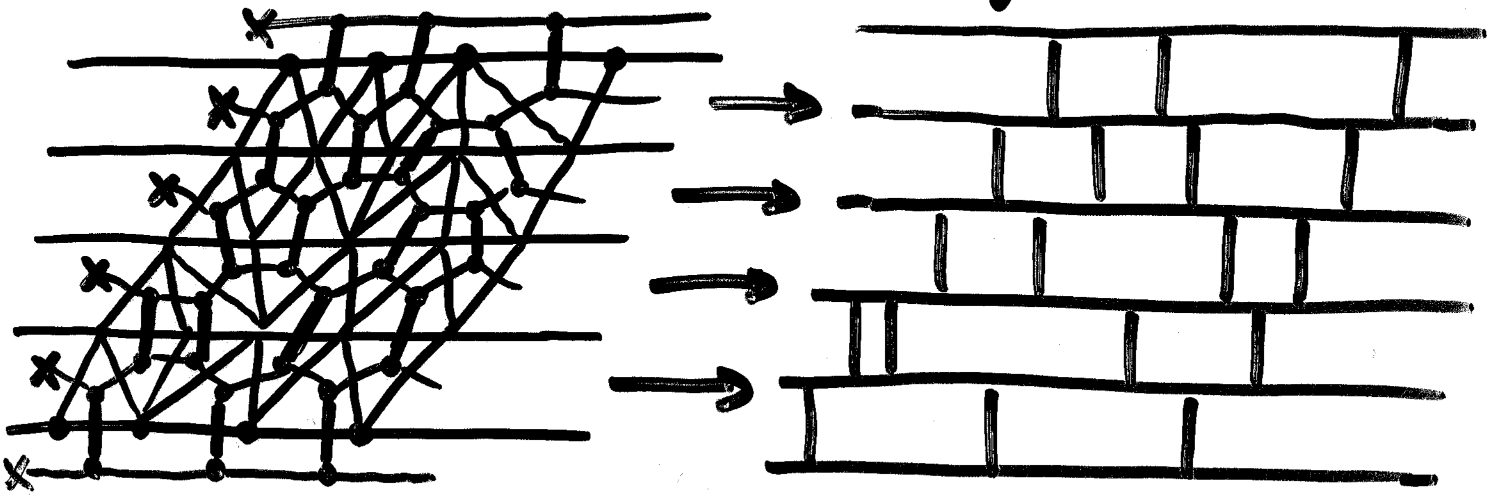
II : Les empilements

Les empilements de dominoes sont en bijection avec les empilement de segments.

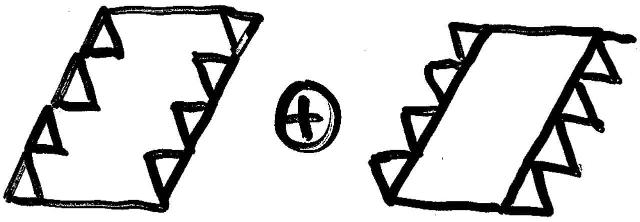


Méthode : matrice de transfer

3



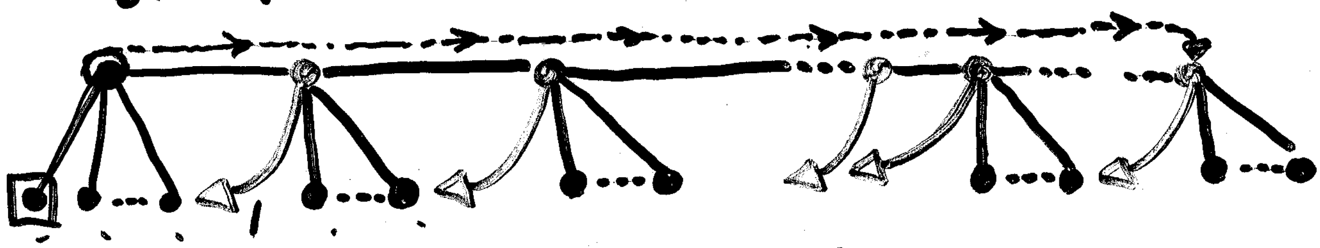
Contraint de borne : escalier aux 2 côtés.



$$\frac{xy(1-q^2)\sqrt{\lambda}^{\,t}}{(1-qx)(1-qy)-\lambda^t(q-x)(q-y)}$$

4

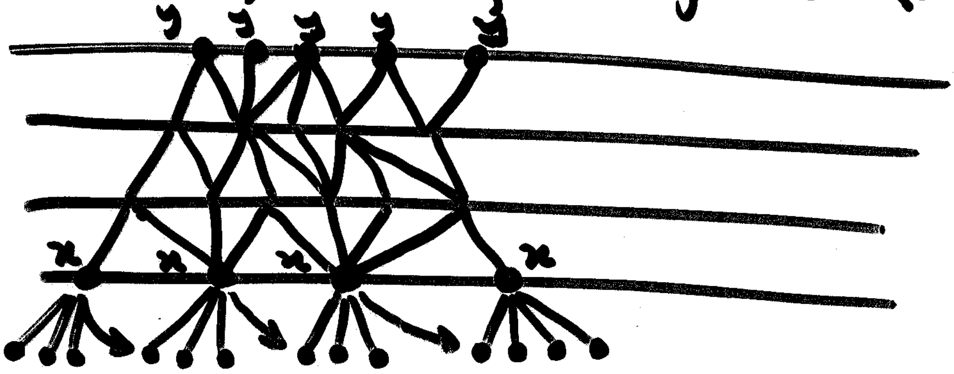
Ambjørn, Loll (cont.)



$$G_t(x,y,g) = \frac{gx}{1-gx}\, G_{t-1}\left(\frac{g}{1-gx}, y, g\right)$$

IV : La Solution Itérative

Modèle de
Di Francesco, Guitter, Kristjansen (avec courbure,



Méthode d'Ambjørn et Loll.

$$G_t(x,y,g) = \frac{gx}{1-gx}\, G_{t-1}\left(\frac{g}{1-gx}, y, g\right)$$

Maintenant :

poid: gx gax ... gax g ga

Avant : $x \longrightarrow \dfrac{g}{1-gx}$

Maintenant : $x \longrightarrow ga + \dfrac{g^2 x}{1-gax}$

$$= \frac{ga + g^2(1-a^2)x}{1-gax}.$$

IV : La solution (réécrite)

$$G_n' = q \frac{1 - xq + \lambda^n/q\,(q-x)}{1 - xq + \lambda^n q\,(q-x)}$$

$G_0 = x$ $\qquad G_n = ga + \dfrac{g^2 G_{n-1}}{1 - ga\,G_{n-1}}$.

$$q = ga + \frac{g^2 q}{1 - gaq}$$

parce qu'on a eu

$$G_n' = \frac{x\,G_n^* + (ga - x)\,G_{n-1}^*}{G_{n-1}^* - g^2 G_{n-2}^*} .$$

Le numérateur <u>et</u> le denominateur satisfient

$$H_n = H_{n-1}\,(1 + g^2(1-a^2)) - H_{n-2}\,g^2$$

ou $H \equiv D$ (resp. N) pour le denominateur (resp. numerateur)

et $D_{n+1} = \prod_{i=0}^{n+1} (1 - ga\,N_i)$, $n \geq 0$

Finalement, on a

$$G_n^* = (1+g^2(1-a^2))^n F_n\left(\frac{g^2}{(1+g^2(1-a^2))^2}\right)$$

$$= \frac{(1-\lambda^{n+1})}{1-\lambda}\left(\frac{1+g^2(1-a^2)}{1+\lambda}\right)^n .$$