Mers de Particules et Séries hypergéométriques

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anujan's $_1\psi_1$ summation $(a)_n = (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$

$$\frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1/a)_n(zqa)^n}{(-bq)_n} = \frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty}$$

auss

$$\sum_{n=0}^{\infty}\frac{(-1/b;q)_n(-1/a;q)_n(cabq)^n}{(q;q)_n(cq;q)_n} = \frac{(-acq;q)_\infty(-cbq;q)_\infty}{(cq;q)_\infty(cabq;q)_\infty}.$$

$a = 0$

$$\frac{1+dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(bc)^n q^{n(n+1)/2}}{(q;q)_n(bdq;q)_n(cdq;q)_n} = \frac{(-dq;q)_\infty(-dcbq}{(dcq;q)_\infty(dbq;q}$$

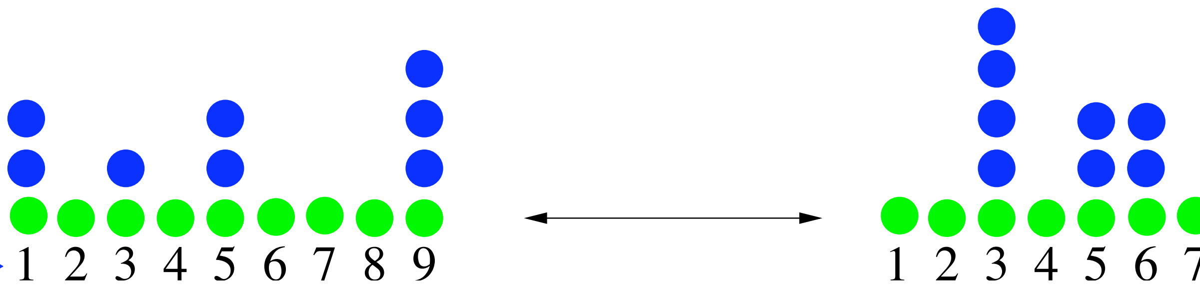
titions

$(\lambda_1, \ldots, \lambda_k)$ $\lambda_1 \geq \ldots \geq \lambda_k$

$= \lambda_1 + \lambda_2 + \ldots + \lambda_k$

$= k$

$9, 5, 5, 3, 1, 1) \leftrightarrow$



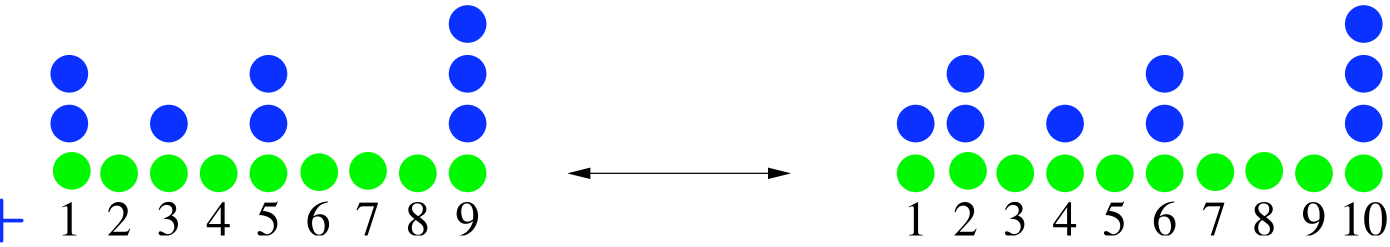
Partitions into parts $\leq n \leftrightarrow$ partitions into $\leq n$ parts

titions

$$\sum_{\lambda \in P} q^{|\lambda|} z^{l(\lambda)} = \prod_{i \geq 1} (1 + zq^i + z^2 q^{2i} + z^3 q^{3i} + \ldots) = \frac{1}{(zq;q)_\infty}$$

$$\sum_{\lambda \in P_n} q^{|\lambda|} z^{l(\lambda)} = \prod_{i=1}^{\infty} (1 + zq^i + z^2 q^{2i} + z^3 q^{3i} + \ldots) = \frac{1}{(zq;q)_n}$$

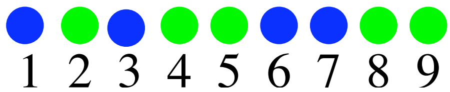
$$\sum_{n} \frac{z^k q^n}{(q;q)_n} = \frac{1}{(zq;q)_\infty}.$$



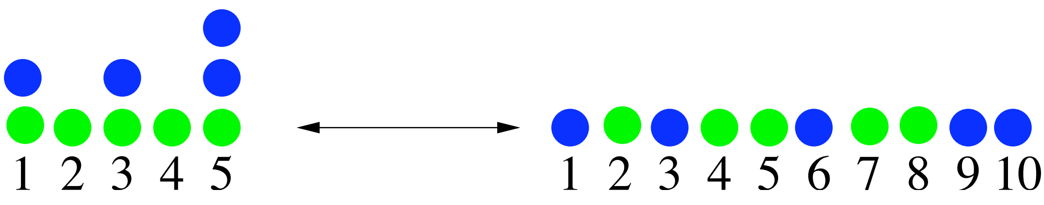
titions into distinct parts

$$\sum_{\lambda\in D} q^{|\lambda|} z^{l(\lambda)} = \prod_{i\geq 1}(1+zq^i) = (-zq;q)_\infty.$$

$$\sum_{\lambda\in D_n} q^{|\lambda|} z^{l(\lambda)} = (-zq;q)_n \qquad \sum_{\lambda\in D_{n,\geq}} q^{|\lambda|} z^{l(\lambda)} = (-z;q)_{n+1}.$$

$3, 1) \leftrightarrow$ 

$$\sum_n \frac{z^k q^{\binom{n+1}{2}}}{(q;q)_n} = (-zq;q)_\infty.$$

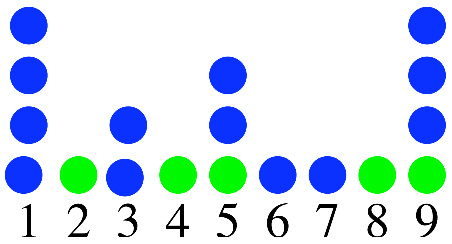
$(5,4,3,2,1)+$ 

erpartitions

rpartition : partition of n in which the last occurrence of rt can be overlined.

$\bar{3})$ $(2,1)$ $(\bar{2},1)$ $(2,\bar{1})$ $(\bar{2},\bar{1})$ $(1,1,1)$ $(1,1,\bar{1})$

$$\sum_{\lambda \in O} q^{|\lambda|} z^{l(\lambda)} = \frac{(-zq;q)_\infty}{(zq;q)_\infty} \qquad \sum_{\lambda \in O_n} q^{|\lambda|} z^{l(\lambda)} = \frac{(-zq;q)_n}{(zq;q)_n}.$$

$(9,9,9,\bar{7},\bar{6},5,5,3,\bar{3},1,1,1,\bar{1}) \leftrightarrow$ 

nomial theorem (Joichi-Stanton)

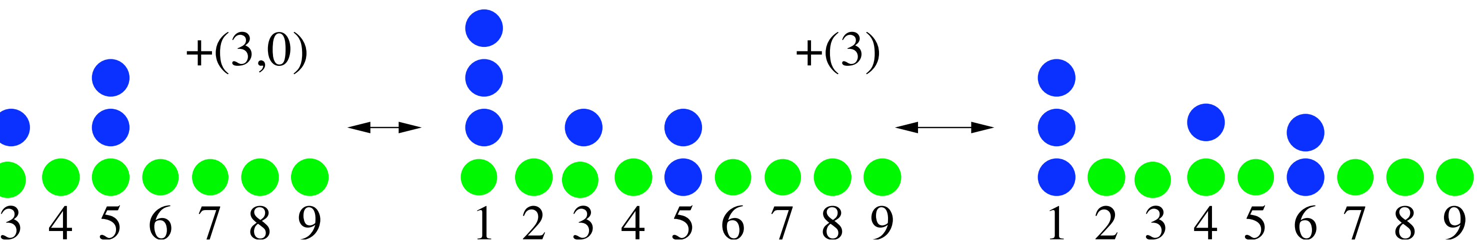
$$\sum_m \frac{(-1/a;q)_m (azq)^m}{(q;q)_m} = \frac{(-zq;q)_\infty}{(zaq;q)_\infty}$$

s : A partition α into m parts and a partition β into nct nonnegative parts less than m.

S : an overpartition

nge α into a particle sea

β : shift i balls to the right. Crash the $(i+1)^{th}$.



nomial theorem (Zeilberger)

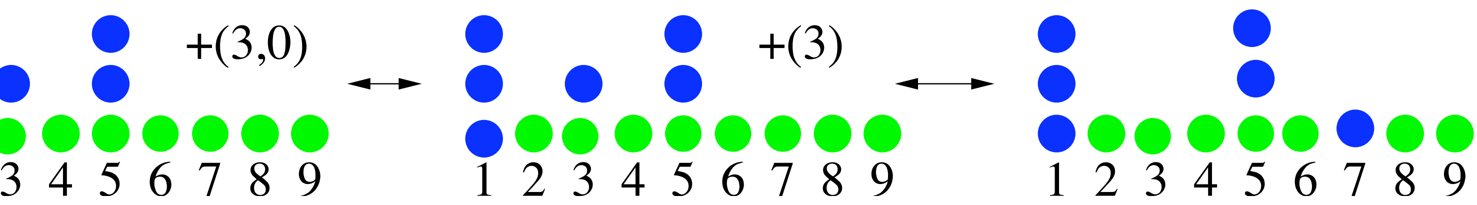
$$\sum_m \frac{(-1/a;q)_m (azq)^m}{(q;q)_m} = \frac{(-zq;q)_\infty}{(zaq;q)_\infty}$$

s : A partition α into m parts and a partition β into nct nonnegative parts less than m.

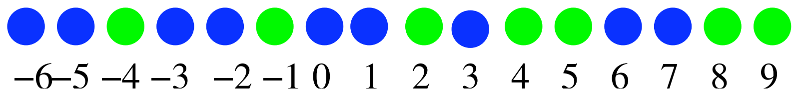
S : an overpartition

nge α into a particle sea

β : Shift the $(i+1)^t h$ ball to the right by i. Crash it.



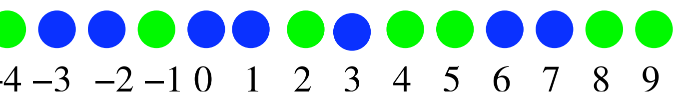
t particle seas : pairs of partitions into distinct parts
e into nonnegative parts)

$3, 2, 0) + (9, 8, 5, 4, 2) \leftrightarrow$ 

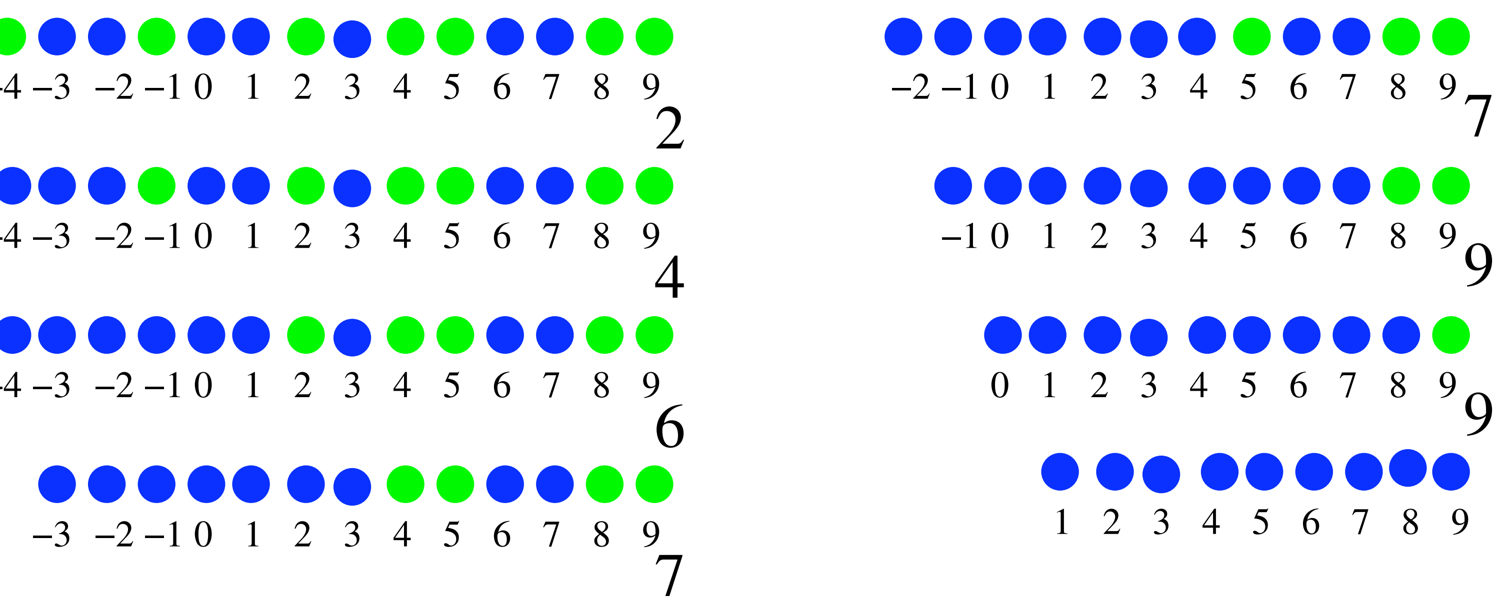
$$\sum F_m(n) q^n z^m = (-zq;q)_\infty (-1/z;q)_\infty$$

obi Triple product

$$(-zq;q)_\infty (-1/z;q)_\infty = \frac{\sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}}}{(q;q)_\infty}$$



$\leftrightarrow (9, 9, 7, 7, 6, 4, 2)$

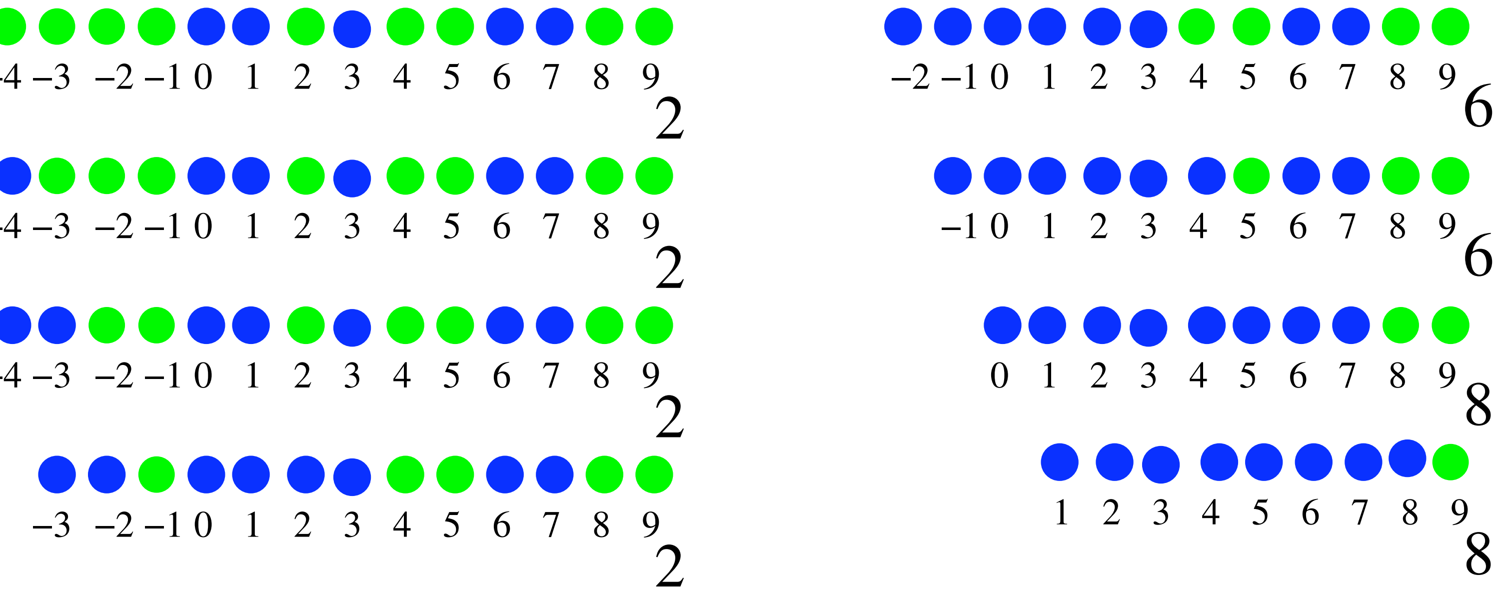


−4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9
2
−4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9
4
−4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9
6
−3 −2 −1 0 1 2 3 4 5 6 7 8 9
7
−2 −1 0 1 2 3 4 5 6 7 8 9
7
−1 0 1 2 3 4 5 6 7 8 9
9
0 1 2 3 4 5 6 7 8 9
9
1 2 3 4 5 6 7 8 9

obi Triple product

$$(-zq;q)_\infty(-1/z;q)_\infty = \frac{\sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}}}{(q;q)_\infty}$$

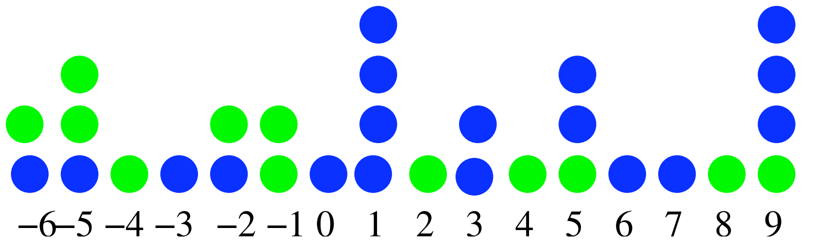
$0) + (9,8,5,4,2)$



$\leftrightarrow (8,8,6,6,2,2,2,2)$ Itzykson, Wright.

ticle seas : pairs of overpartitions (one into ≥ 0 parts)

$9, \bar{7}, \bar{6}, 5, 5, 3, \bar{3}, 1, 1, 1, \bar{1}) + (6, 5, 5, \bar{4}, \bar{2}, 1, \bar{1}) \leftrightarrow$



$$\sum S_m(n,k,l) q^n a^k b^l z^m = \frac{(-zq;q)_\infty(-1/z;q)_\infty}{(zaq;q)_\infty(b/z;q)_\infty}$$

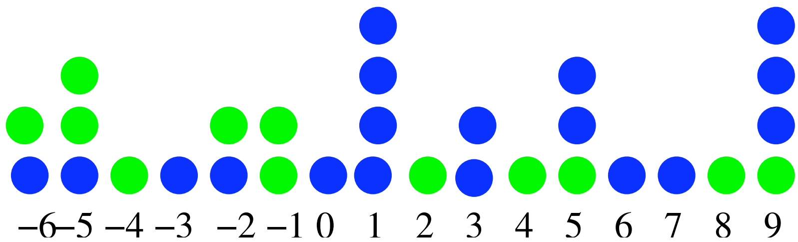
manujan's ${}_1\psi_1$ summation

$$\frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty} = \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1/a)_n(zqa)^n}{(-bq)_n}$$

$$\frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty} = \sum_{m,n} \frac{(-1/b)b^m}{(q)_m}(1/a)_n(aq)^n(q)_n$$

pret each summand as a particule sea with m green balls on the
ositive side and n blue balls on the positive side.

ht = abscissas of the green balls of the nonpositive side + abscissas
e blue balls of the positive side.

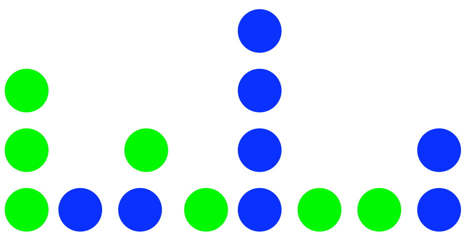


$_1\psi_1$

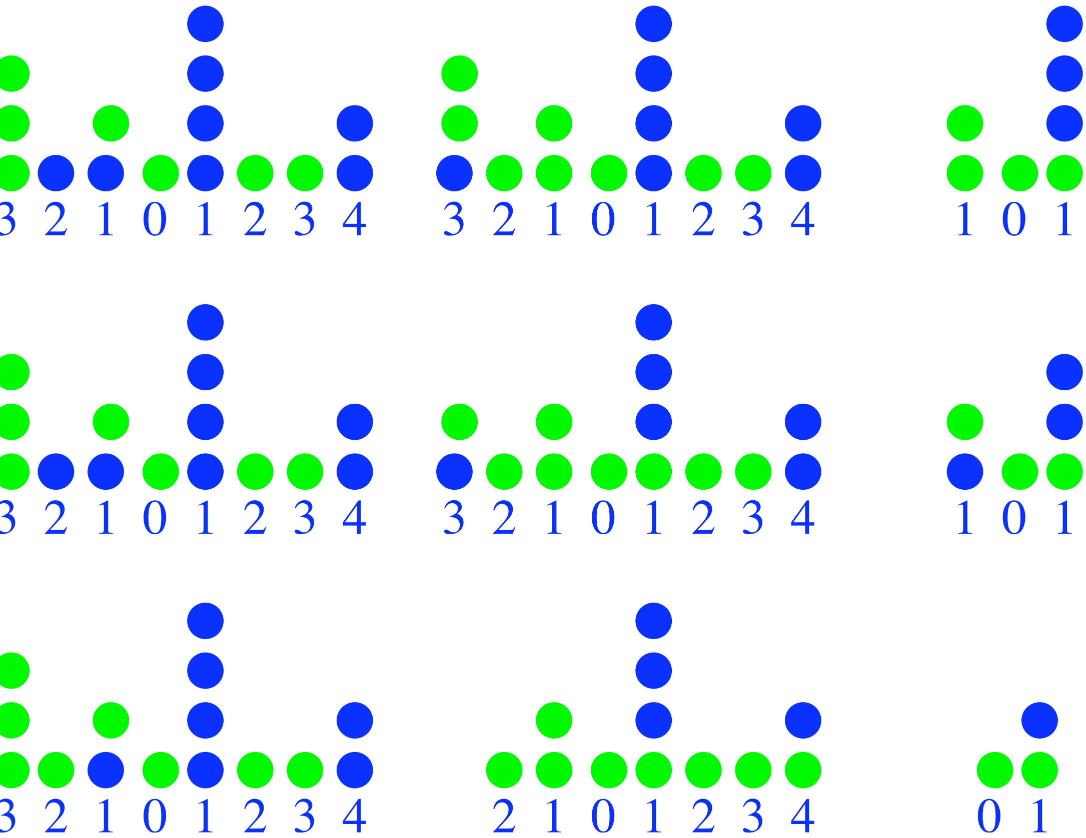
$$[z^0]\frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty} = \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty}$$

auss

$$\sum_n \frac{(-1/a)_n(-1/b)_n}{(q)_n(q)_n}(abq)^n = \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty}.$$



$3\ 2\ 1\ 0\ 1\ 2\ 3\ 4 \leftrightarrow (\bar{7},\bar{4},2,2,\bar{1}) + (\bar{6},2,\bar{2},\bar{1}).$



$$\leftrightarrow (\bar{7}, \bar{4}, 2, 2, \bar{1}) + (\bar{6}, 2, \bar{2}, \bar{1}).$$

manujan's ${}_1\psi_1$ summation

$$\frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty} = \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1/a)_n(zqa)^n}{(-bq)_n}$$

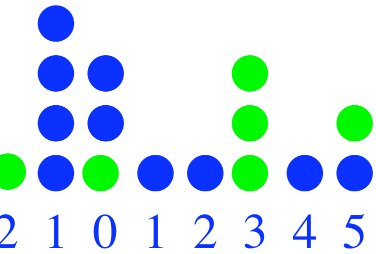
$$A_n = [z^n]\frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty}$$

$$A_0 = \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty} \qquad A_n = A_0\frac{(-1/a)_n(zqa)^n}{(-bq)_n}$$

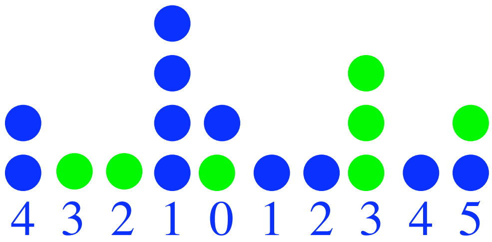
$$0 \;\; A_0\prod_{i=0}^{n-1}(b+q^i) = A_{-n}\prod_{i=1}^{n}(1+aq^i); \;\; A_0q^n\prod_{i=1}^{n}(a+q^{i-1}) = A_n\prod_{i=1}^{n}(1+$$

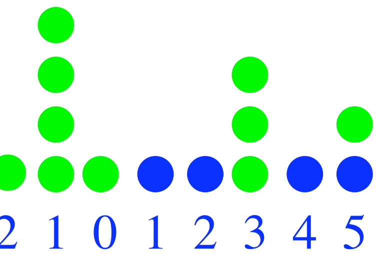
ample $n = -4$

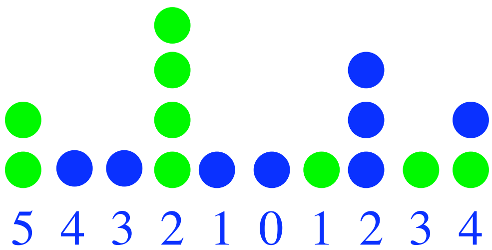
$A_{-4} \prod_{i=1}^{4} (1 + aq^i) = A_0 \prod_{i=0}^{3} (b + q^i)$



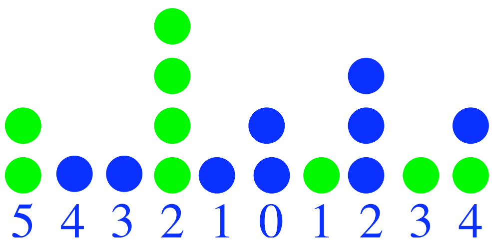
(3)







(2)



(3,2)

ther? $6\phi5$

$$\sum_{n=0}^{\infty} \frac{(1+dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(-d/a;q)_n(abcq)^n}{(q;q)_n(bdq;q)_n(cdq;q)_n(aq;q)_n} =$$

$$\frac{(-acq;q)_\infty(-abq;q)_\infty}{(aq;q)_\infty(cabq;q)_\infty}\frac{(-dq;q)_\infty(-dcbq;q)_\infty}{(dcq;q)_\infty(dbq;q)_\infty}.$$

0 : q-Gauss

0 :

$$\frac{1 + dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(bc)^n q^{n(n+1)/2}}{(q;q)_n(bdq;q)_n(cdq;q)_n} = \frac{(-dq;q)_\infty(-dcbq}{(dcq;q)_\infty(dbq;q}$$

ored particle seas

k squares appear in the positive quarter and do not appear on the line

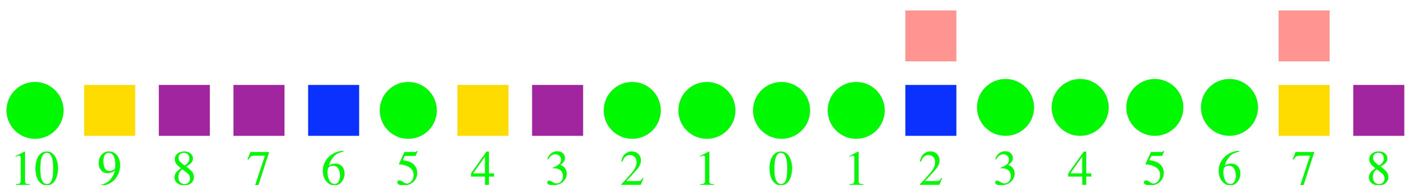
een particules and yellow, purple and blue squares appear on the zero

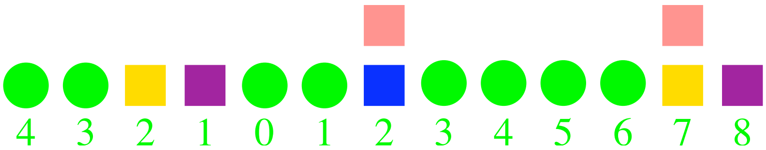
o blue squares do not appear consecutively.

tween two green balls, squares can be yellow then purple then blue.

mber of squares in the positive quarter = number of balls in the ositive quarter.

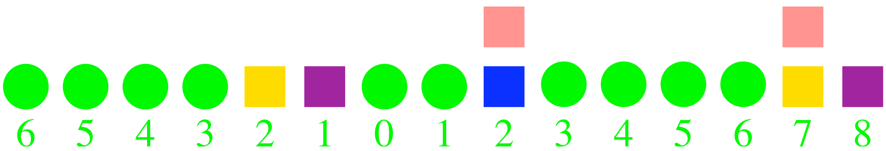
ht : abscissas of the green particules in the nonpositive quarter and sses of the squares in the positive quarter.





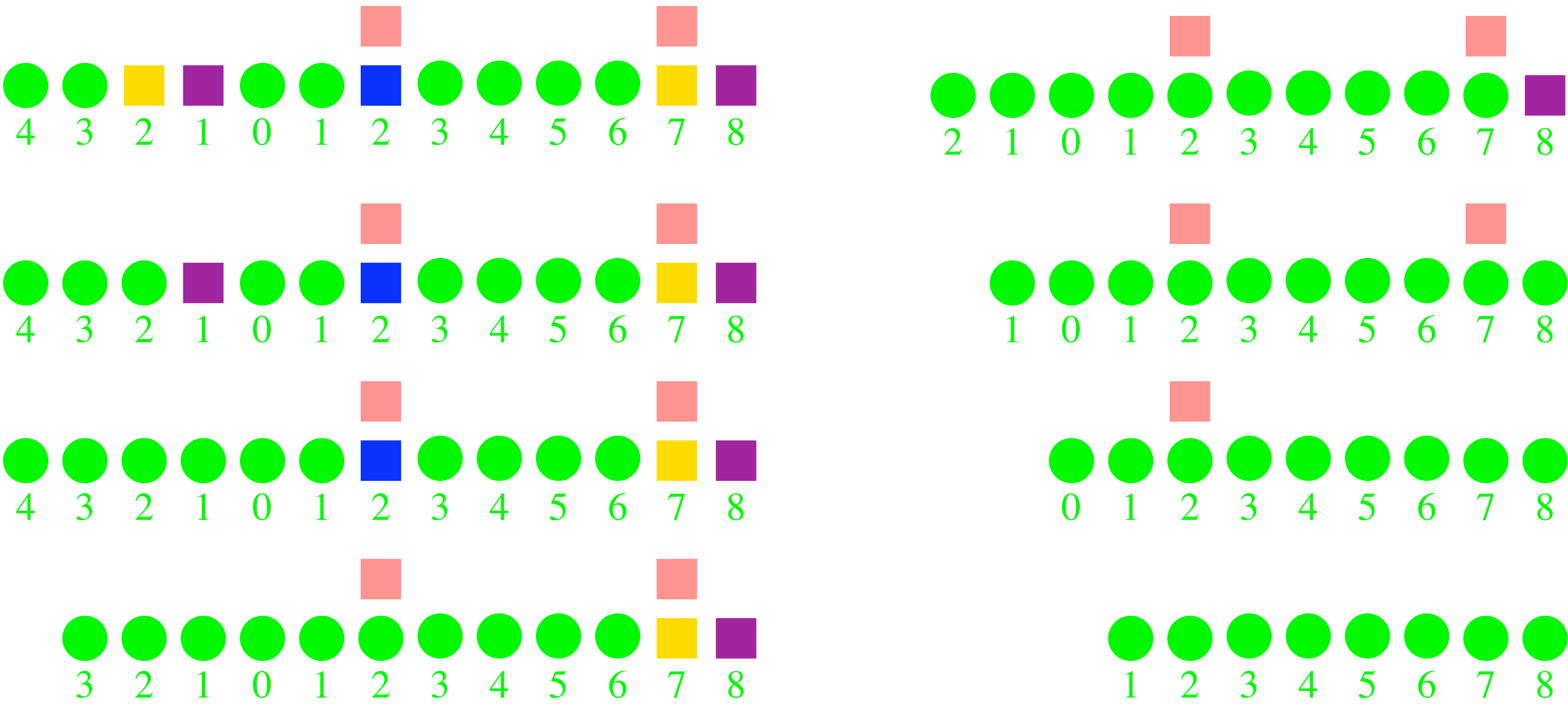
$$\sum_{s\in C} q^{|s|} b^{\mathrm{pink}(s)+\mathrm{purple}(s)} c^{\mathrm{pi}(s)+\mathrm{blue}(s)}$$

$$= \sum_n \frac{(-q;q)_{n-1}(1+q^{2n})q^{n(n-1)/2}}{(bq;q)_n(cq;q)_n} \frac{(-1/b;q)_n(-1/c;q)_n(qbc)^n}{(q;q)_n}.$$



$\leftrightarrow (10, \bar{10}, 4, \bar{4}) + (\bar{8}, 6, \bar{2})$

$$\frac{1 + dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(bc)^n q^{n(n+1)/2}}{(q;q)_n(bdq;q)_n(cdq;q)_n} = \frac{(-dq;q)_\infty(-dcbq}{(dcq;q)_\infty(dbq;q}$$



$$\leftrightarrow (10, \overline{10}, 4, \bar{4}) + (\bar{8}, 6, \bar{2})$$

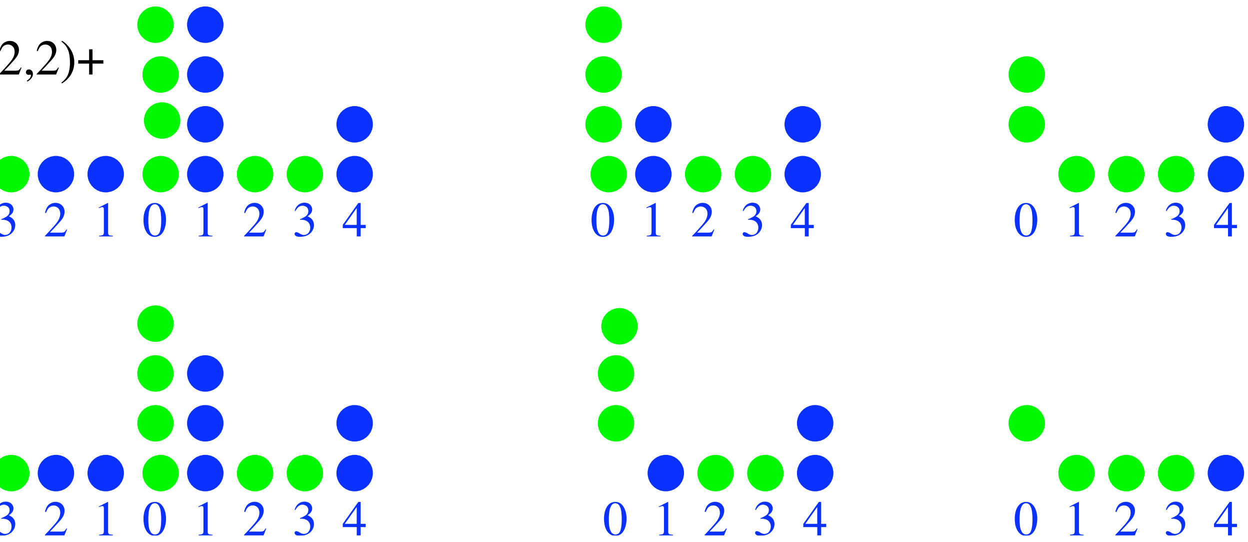
v results (Yee)

$$\frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty} = \sum_{m,n} \frac{(-1/b)b^m}{(q)_m}(1/a)_n(aq)^n(q)_n$$

n Interpret each summand as a partition into parts $\leq m$ and a
cule sea m green (resp. n blue) balls on the nonpositive (resp.
ive) side. Nonpositive side : all the balls not on the zero line have
abscissa.

mple : $m = n = 6$

2,2)+



$$\leftrightarrow (\bar{6}, 4, \bar{3}, 1, 1) + (4, \bar{4}, 3, 2, 2)$$

aff $\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1/a,-1/b;q)_m}{(cq,cq^{n-m+1}ab;q)_m} (cabq)^m = \frac{(-caq,-cbq;q)_n}{(cq,cabq;q)_n}$

ect interpretation for ${}_1\psi_1$

$$\frac{(-aq)_\infty}{(q)_\infty(abq)_\infty}\sum_{n=0}^{\infty}(-1/a)_n(-bq^{n+1})_\infty(zqa)^n +$$
$$\frac{(-bq)_\infty}{(q)_\infty(abq)_\infty}\sum_{n=1}^{\infty}(-1/b)_{-n}(-aq^{-n+1})_\infty(z/b)^{-n} = \frac{(-zq)_\infty(-z^{-1})_\infty}{(bz^{-1})_\infty(azq)_\infty}$$

off $m-n$ green balls from the nonpositive side : all the k balls on ero line that are $\leq m-n-1$ $+m-n-k$ balls not on the zero line.