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Analytic Urns

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General: BALLS and one or more URNS.

Two kinds of models

• "Balls-and-bins": Throw balls at random into a number of urns.

= Random allocations. Basic in the analysis of hashing algorithms; also SAT problem, cf V. Puyhaubert.

= Techniques: Exponential generating functions and saddle point. Poissonization &c.
Kolchin et al., *Random Allocations*, 1978.

• "Urn models": One urn contains balls whose nature may randomly change according to ball drawn and finite set of rules.

Here: URNS with BALLS of TWO COLOURS

Type I	Type II
Black	White

RULES are given by a 2×2 Matrix

The composition of the urn at time 0 is fixed. At time n, a ball in the urn is randomly chosen and its colour is inspected (thus the ball is drawn, looked at and then placed back in the urn): if it is black, then α black and β white balls are subsequently inserted; if it is white, then, γ black balls and δ white balls are inserted.

drawn $\downarrow$	added B	added W
B	α	β
W	γ	δ .

- Drawing with replacement = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- Drawing without replacement = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

- Laplace's "melancholic" model (1811): if a ball is drawn, it is repainted black no matter what its colour is.

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

- Ehrenfest & Ehrenfest = *Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem,* 1923. Irreversibility contradicts Ergodicity.

Exchanges of basic balls ("atoms of heat") between two urns, one cold and one hot $\leadsto$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Bernoulli (1768), Laplace (1812).

• Pólya Eggenberger model. A ball is drawn at random and then replaced, together with s balls of the same colour.

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

A model of positive influence. Closed form.

• "Adverse influence" model

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$$

Used in epidemiology, etc.

• The special search tree model

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$$

Yao (1978); Bagchi and Pal (1985); Aldous et al (1988); Prodinger & Panholzer (1998)

Johnson & Kotz, *Urn Models and their Application*, Wiley 1973

Here case of a 2×2-matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with constant row-sum $\heartsuit\heartsuit$

$$s := \alpha + \beta = \gamma + \delta.$$

At time n size t_n satisfies $t_n = t_0 + sn$.

Constant increment s

A problem with three parameters + two initial conditions.

♠ Kotz, Mahmoud, Robert (2000) show "pathologies' in some of the other cases.

Huge literature: *Math. Reviews*

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Lead to amazingly wide variety of behaviours, special functions, and limit distributions.

Methods

- Difference equations and explicit solutions.
- Same but with probability generating functions.
- Connection with branching processes.
- Stochastic differential equations (KMR)
- Martingales (Gouet)

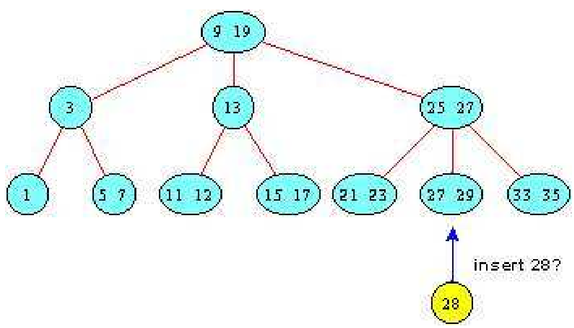
Here: A frontal attack:
— PDE of snapshots at time n
— Usual solution for quasilinear PDE
— Bivariate GF and singularity perturbation
Conformal mapping argument, Abelian integrals over Fermat curves $z^h + y^h = 1$
+ Special solutions with elliptic functions

Part I

The $\mathcal{T}_{2,3}$ model—basic equations

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$$

• Insertions in a 2–3 tree: 2–node $\mapsto$ 3–node; 3–node $\mapsto$ (2–node + 2–node).



• Fringe-balanced 2–3 tree analogous to median-of-three quicksort.

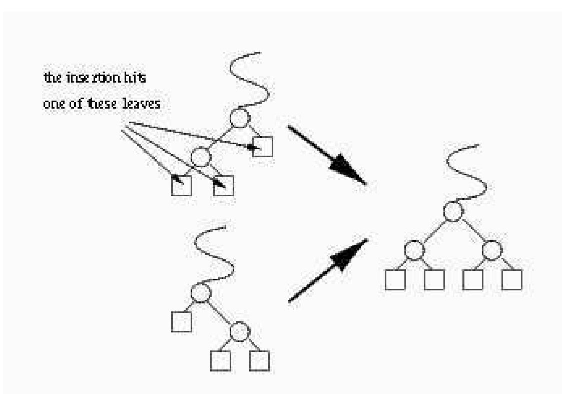


FIGURE 1. The fringe heuristic

Mahmoud (1998); Panholzer–Prodinger (1998)

Evolution is:

$$X_1 = 2; \qquad X_n - X_{n-1} = \begin{cases} -2 & \text{with probability } \frac{X_{n-1}}{n} \\ +4 & \text{with probability } 1 - \frac{X_{n-1}}{n}. \end{cases}$$

Let $p_{n,k} = \mathbb{P}(X_n = k)$, $p_n(u) := \sum_k p_{n,k} u^k$, and

$$F(z,u) := \sum_{n \geq 1} p_n(u) u^n = \sum_{n,k} p_{n,k} u^k z^n,$$

whose elicitation is our main target.

Lemma: PDE satisfied by BGF of probabilities is

$$(u^5 z - u)\frac{\partial F}{\partial z} + (1 - u^6)\frac{\partial F}{\partial u} + u^5 F + u^3 = 0.$$

Proof. Each p_n is determined from previous one by ∂_u = a differential recurrence. Gives PDE for bivariate generating function F.

Take $p_0(u)$ that satisfies PDE and write $G := p_0(u) + F(z,u)$. then, we get a *homogeneous PDE*.

$$(u^5 z - u)\frac{\partial G}{\partial z} + (1 - u^6)\frac{\partial G}{\partial u} + u^5 G = 0.$$

with

$$p_0(u) = (1 - u^6)^{1/6} \int_0^u t^3 (1 - t^6)^{-7/6}\, dt.$$

Quasilinear first-order PDE's are reducible to ODEs.

$$A(z,u,G)\frac{\partial G(z,u)}{\partial z}+B(z,u,G)\frac{\partial G(z,u)}{\partial u}+C(z,u,G)=0$$

1. Look for a solution in implicit form $X(z,u,G)=0$.

$$A(z,u,w)\frac{\partial X}{\partial z}+B(z,u,w)\frac{\partial X}{\partial u}-C(z,u,w)\frac{\partial X}{\partial w}=0.$$

2. Consider the ordinary differential system

$$\frac{dz}{A}=\frac{du}{B}=-\frac{dw}{C}.$$

The solution of two "independent" ordinary differential equations, e.g.,

$$\frac{du}{B}=-\frac{dw}{C} \quad \text{and} \quad \frac{dz}{A}=\frac{du}{B},$$

leads to two families of integral curves,

$$U(u,z,w)=C_1 \quad \text{and} \quad V(u,z,w)=C_2.$$

3. The generic solution of the PDE is provided by

$$X(z,u,w)=\Phi(U(u,z,w),V(u,z,w)),$$

for arbitrary bivariate Φ. Solving for w in $X(z,u,w)=0$ provides a relation $w=R_\Phi(z,u)$. General solution is

$$G(z,u):=R_\Phi(z,u).$$

$$(u^5 z - u)\frac{\partial G}{\partial z} + (1-u^6)\frac{\partial G}{\partial u} + u^5 G = 0.$$

Consider

$$\frac{du}{1-u^6} = \frac{dz}{u^5 z - u} = -\frac{dw}{u^5 w}.$$

- $du \leftrightarrow dw$ first integral by separation:

$$w(1-u^6)^{-1/6} = C_1.$$

- $du \leftrightarrow dz$ variation of constant:

$$z(1-u^6)^{1/6} + \int_0^u \frac{t}{(1-t^6)^{5/6}}\, dt = C_2.$$

Bind the two integrals by arbitrary Φ & $w \equiv G$

$$\Phi\left(\frac{G}{(1-u^6)^{1/6}},\ z(1-u^6)^{1/6} + \int_0^u \frac{t}{(1-t^6)^{5/6}}\, dt\right) = 0,$$

Solve for G, introducing arbitrary ψ:

$$G(z,u) = \delta(u)\psi(\delta(u)z + I(u)), \quad I(u) := \int_0^u \frac{t}{(1-t^6)^{5/6}}\, dt,$$

with $\delta(u) := (1-u^6)^{1/6}$.

Initial conditions identify ψ.

Theorem 1. *Define the quantities*

$$\delta(u) = (1-u^6)^{1/6},$$
$$I(u) = \int_0^u \frac{t}{(1-t^6)^{5/6}}, \quad J(u) = \int_0^u \frac{t^3}{(1-t^6)^{7/6}}\,dt. \tag{1}$$

Then, the bivariate generating function of the probabilities is

$$G(z,u) = \delta(u)\psi\left(z\delta(u) + I(u)\right), \tag{2}$$

where ψ is the function defined parametrically for $|u| < 1$ by

$$\psi(I(u)) = J(u). \tag{3}$$

Dominant singularities of ψ?

The diagram that summarizes ψ is

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ z = I(u) & \xrightarrow{\psi} & \psi(z) = J(u). \end{array}$$

The radius of analyticity of ψ is

$$\rho = I(1), \qquad I(u) := \int_0^u \frac{t}{(1-t^6)^{5/6}}\, dt,$$

Proof: There is local (analytic) invertibility of $I(u)$ along $(0,\rho)$. Thus ψ is analytic along $(0,\rho)$.

We have $\psi(z) = G(z,0)$ which has nonnegative coeffs and is Pringsheim.

We have $I(1) < \infty$ while $J(1) = \infty$. Thus ρ is a singularity.

By Eulerian Beta integrals:

$$\rho \equiv I(1) = \frac{1}{6} B(\frac{1}{6}, \frac{1}{3}) = \frac{1}{6} \frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} \doteq 1.40218\,21053\,25454.$$

$$\left(\frac{\psi_{47}}{\psi_{50}}\right)^{1/3} \doteq 1.40218\,21053\,2545\underline{6},$$

Local expansions near $u = 1$ plus symmetries of the problem are compatible with:

Proposition 1. *There are no singularities of $\psi(z)$ on $|z| = \rho$ other than ρ, $\rho\omega$, $\rho\omega^2$ that are simple poles. Precisely, let*

$$S(z) = \frac{1}{\rho - z} + \frac{1}{\rho\omega - z} + \frac{1}{\rho\omega^2 - z} = \frac{3z^2}{\rho^3 - z^3}.$$

The function

$$\psi(z) - S(z)$$

is analytic in a disc $|z| < R$ for some R satisfying $R > \rho$. (One can take $R = 2\rho$.)

Why singularities of ψ, BTW?

$$G(z, u) = \delta(u)\psi(z\delta(u) + I(u)), \qquad \psi(I(u)) = J(u).$$

Set $u = 0$ and estimate $[z^n]\psi(z)$: Get extremely large deviations, all balls of one colour.

Know approximately $[z^n]G(z, u)$ = PGF of distribution $\rightsquigarrow$ LIMIT LAW.

Set u to value $\neq 1$ and get LARGE DEVIATIONS.

And a good deal more. . .

Part II

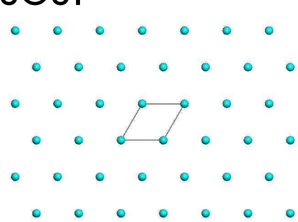
The $\mathcal{T}_{2,3}$ model—elliptic structure

Recall: an elliptic function is a doubly periodic meromorphic function in $\mathbb{C}$.

Historically: Integration over a conic $\int Q(z, y)$ where $y = \sqrt{P(x)}$ and $\deg P = 1, 2$, yields functions like $\arctan$, $\arcsin$ and hyperbolic counterparts. Such functions satisfy $\arctan(z) \cong \arctan(z) + k\pi$ so that inverses are simply periodic. This is a way to (re)build trigonometry from integrals over conics.

Integration over a cubic or a quartic $y = \sqrt{P(x)}$ with $\deg P = 3, 4$, which are topologically "doughuts" leads to double periodicity. Such things occur when rectifying the ellipse hence the name elliptic integrals and elliptic functions for inverses.

$\sum 1/(z - \omega)^3$



For a parameterized curve, $\psi(I(u)) = J(u)$, examine all possible paths in the u-plane, and the corresponding determinations of $I(u)$. Reflect on

$$\psi\left(\int_1^u \frac{dt}{t}\right) = u,$$

which defines $\psi(\log u) = u$, that is, $\psi(z) = e^z$.

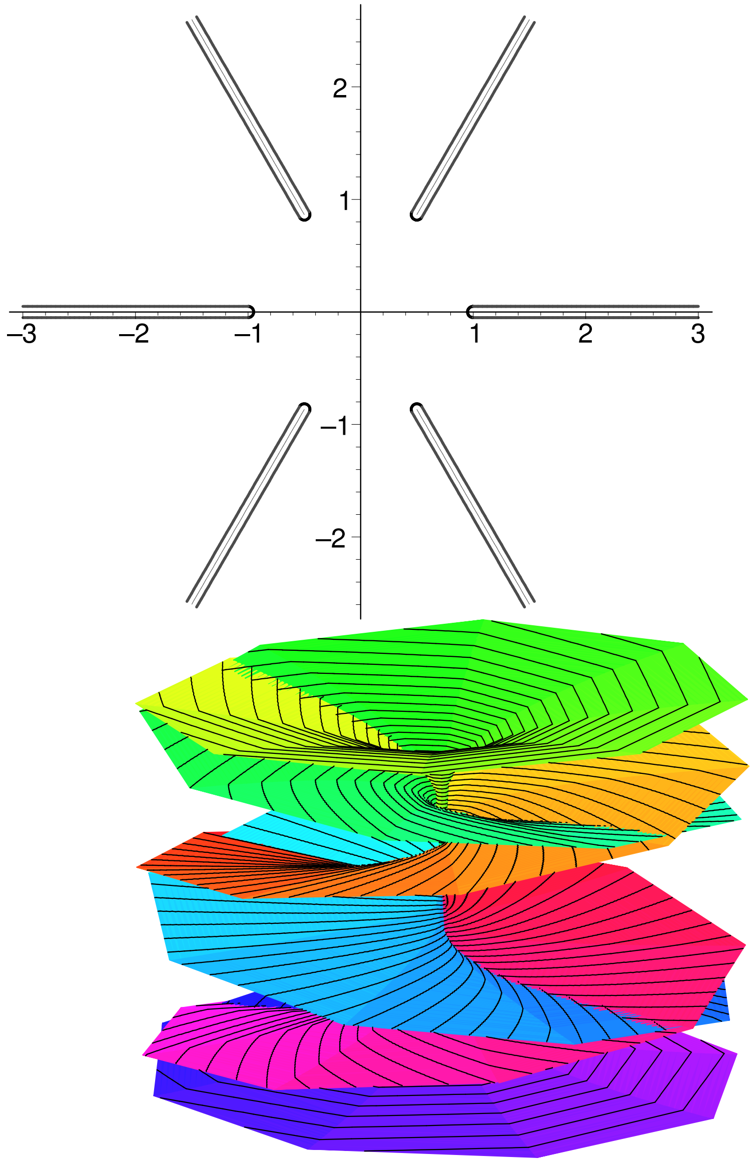
Here:

$$\psi(I(u)) = J(u)$$
$$I(u) = \int_0^u \frac{t\,dt}{(1-t^6)^{5/6}}, \quad I(u) = \int_0^u \frac{t^3\,dt}{(1-t^6)^{7/6}}.$$

The curve is $t^6 + y^6 = 1$ and has genus 10.

Go step by step.

- **The elementary triangle**
- **The fundamental triangle**

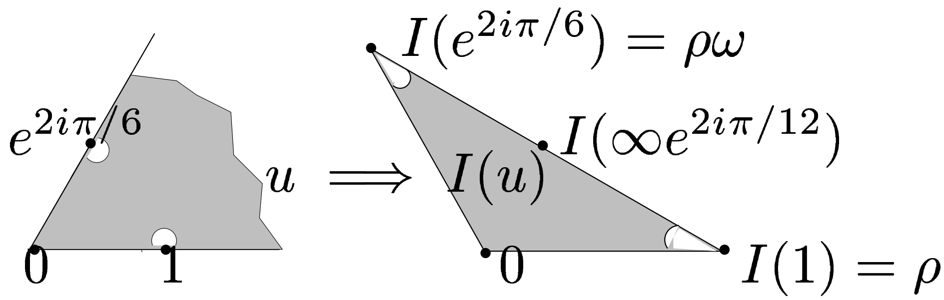


The region R_0 (left) and a rendering of the six-sheeted Riemann surface $\mathfrak{R}$ of $\delta(u) \equiv (1-u^6)^{1/6}$ for u near 1 (right).

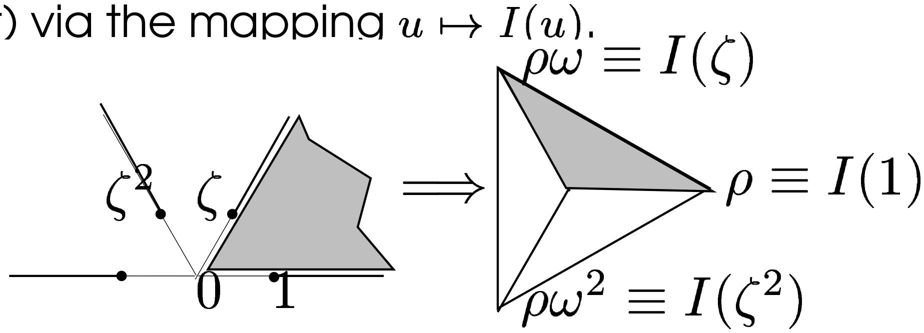
Because of double parameterization, taking u in a half-plane suffices.

Lemma 1. *The function ψ maps the interior of $(R_0 \cap H)$ in a one-to-one manner on the interior of the equilateral triangle T with vertices $\rho, \rho\omega, \rho\omega^2$, where $\omega := e^{2i\pi/3}$.*

Proof. Folds angles in an appropriate way...
Start with Elementary triangle.

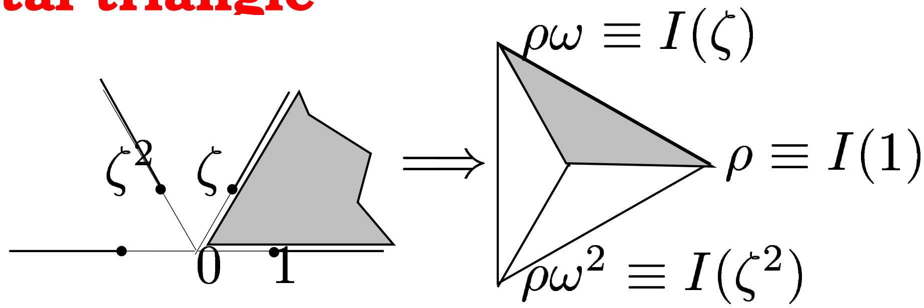


The "elementary triangle" T_0 (right) is the image of the basic sector S_0 (left) via the mapping $u \mapsto I(u)$.

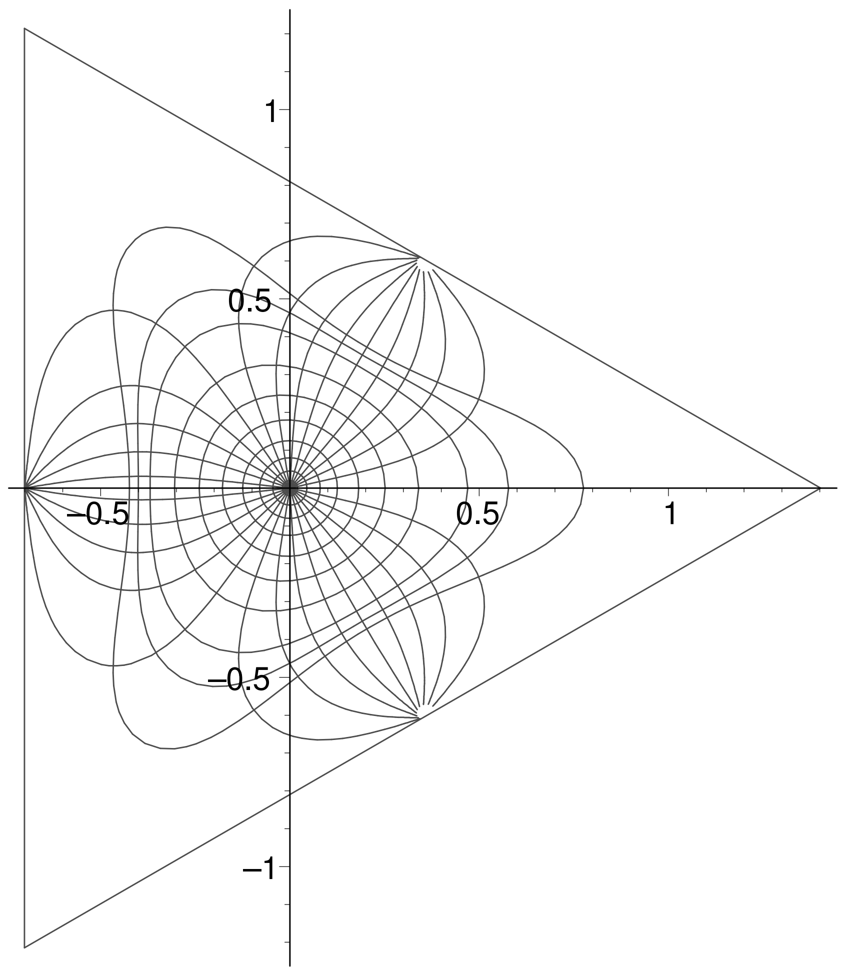


The "fundamental triangle" T (right) is the image of the slit upperhalf plane $(R_0 \cap H)$ (left) via the mapping $u \mapsto I(u)$.

Three **elementary triangles** assemble to form a **fundamental triangle**

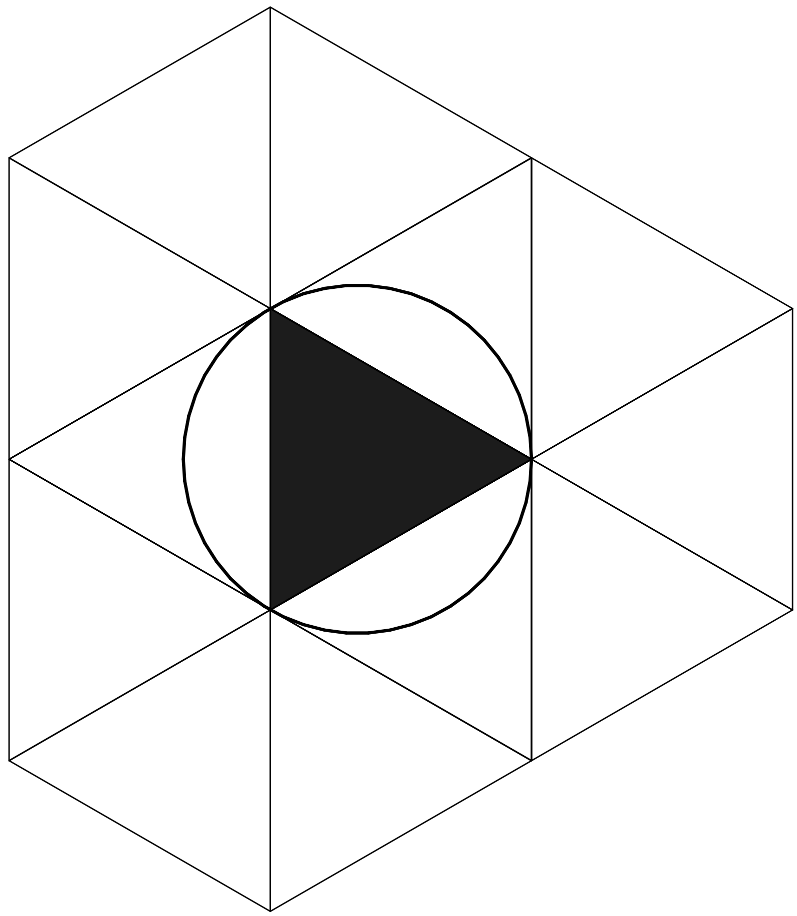


Based on $I(\zeta u) = \omega I(u)$, where $\zeta^6 = 1$, $\omega^3 = 1$.



Another view of the image of $(R_0 \cap H)$ by $I(u)$ giving the fundamental triangle T: a representation of the images of rays emenating from 0 and of circles centred at 0

Lemma 2. *The function ψ is analytic (holomorphic) in the disk $|z| < 2\rho$ stripped of the points $\rho, \rho\omega, \rho\omega^2$. (The function admits these three points as simple poles, as asserted in Prop. 1.)*



Rotated copies of the fundamental triangle around $\rho, \rho\omega, \rho\omega^2$ shown against the circle of convergence of $\psi(z)$.

Proof. Laces around $u = 1$ and changes of variables: $I(1) - I(u) \sim 6^{1/6}(1-u)^{1/6}$.

The full story and the elliptic connection

A *lattice* Λ with generators $\xi, \eta \in \mathbb{C}$:

$$\Lambda(\xi, \eta) = \left\{ n_1\xi + n_2\eta \;\middle|\; n_1, n_2 \in \mathbb{Z} \right\}.$$

The Weierstraß *zeta function* relative to Λ is classically defined as

$$\zeta(z; \Lambda) := \frac{1}{z} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

Theorem 2. *The ψ-function of the $\mathcal{T}_{2,3}$ model intialized with 2 balls of the first type ($a_0 = t_0 = 2$) is exactly*

$$\psi(z) = \frac{1}{\rho\sqrt{3}} \left(-\zeta\left(\frac{z-\rho}{\rho\sqrt{3}} \right) + \zeta\left(-\frac{1}{\sqrt{3}} \right) \right), \quad \rho := \frac{1}{6} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})}, \tag{4}$$

where $\zeta(z) := \zeta(z; \Lambda_{\text{hex}})$ is the Weierstraß zeta function of the hexagonal lattice:

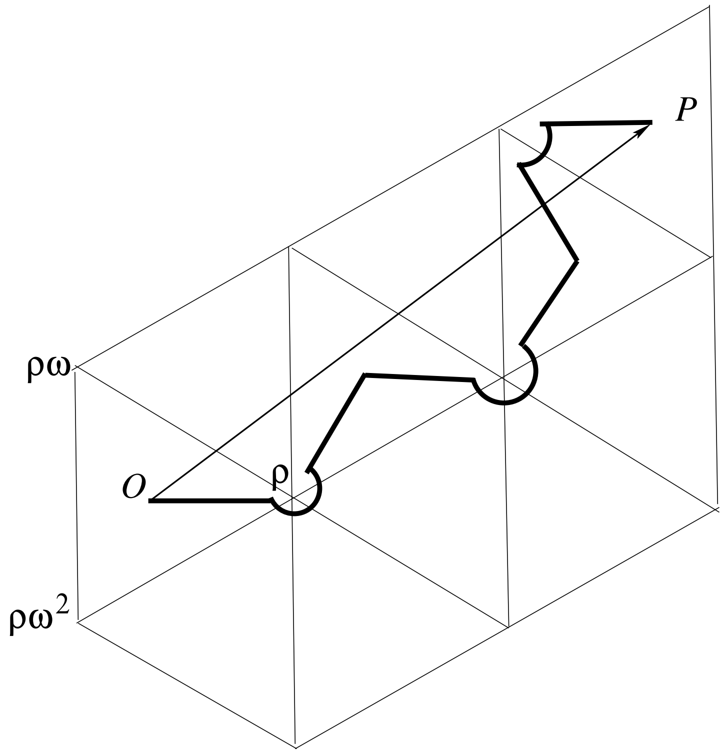
$$\Lambda_{\text{hex}} := \left\{ n_1 e^{i\pi/6} + n_2 e^{-i\pi/6} \;\middle|\; n_1, n_2 \in \mathbb{Z} \right\}.$$

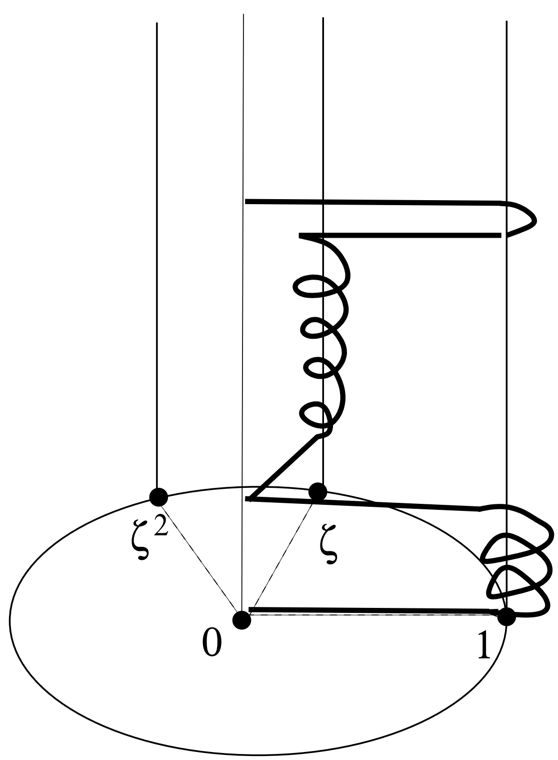
Proof.

• Follow all paths and examine $I(\gamma(u))$: any point $z \in \mathbb{C}$ is reachable.

• There is a pole of ψ at lattice points and residue is -1 since determinations of $\delta(u)$ in I, J are the same.

• By Liouville, $\psi(z)$ and ζ coincide (up to normalization).





A path in the z-plane from 0 to P and the contour γ above the u-plane that realizes it via $u \mapsto z = I(u)$.

Part III

Probabilistic consequences

Extract coeffs in simple fractions:

Corollary 1. *For the $\mathcal{T}_{2,3}$ model, the probability generating function $p_n(u) = \mathbb{E}(u^{X_n})$ admits an* exact *formula valid for all $n \geq 2$,*

$$p_n(u) = \sum_{n_1,n_2=-\infty}^{+\infty} \left(K(u) + \frac{\rho\sqrt{3}}{\delta(u)}(n_1 e^{i\pi/6} + n_2 e^{-i\pi/6}) \right)^{-n-1},$$

where

$$K(u) := \frac{1}{\delta(u)} \int_u^1 \frac{t}{\delta(t)^5}\, dt, \qquad \delta(u) = (1-u^6)^{1/6}.$$

Note: when $u \approx 1$, this is like $K(u)^{-n-1}$.

The Quasi-powers framework.

Classics are:

(Laplace) Given a random variable X, define its characteristic function aka **Fourier transform** as

$$\phi_X(t) := \mathbb{E}(e^{itY}) = \sum_k \mathbb{P}(Y = k)e^{itk} = p(e^{it}).$$

If $S_n = X_1 + \cdots + X_n$ with i.i.d. X_j, then:

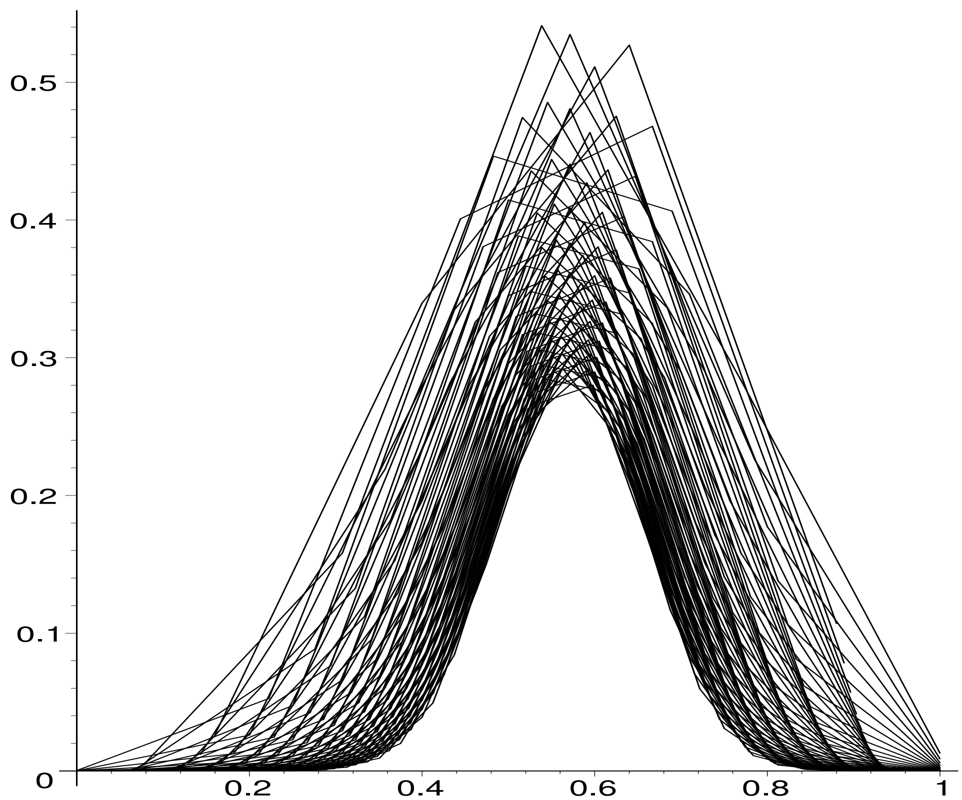
$$\phi_{S_n}(t) = (\phi_X(t))^n .$$

(Lévy et al.) Fourier inversion is continuous: convergence of F.T.'s

$$\lim_{n\to\infty} \phi_{Y_n}(t) = \phi_Z(t) \quad \text{pointwise}$$

implies $Y_n \Rightarrow Z$ in distribution.

(Berry-Esseen) Uniform distance on F.T. furthermore gives bounds on uniform distance on distribution functions.



A Sedgewick plot of $\{\mathbb{P}(x_n = k)\}_{k=0}^{n-1}$ for $n = 24 \mathinner{..} 96$ (the horizontal axis is normalized to $n+1$).

Gaussian laws in analytic combinatorics

Classically $S_n = X_1 + \cdots + X_n$, where X_j have mean and variance. Calculation shows that

$$\log \mathbb{E}\left[\exp\left(it\frac{S_n - n\mu}{\sigma\sqrt{n}}\right)\right] \underset{n\to\infty}{\longrightarrow} -\frac{t^2}{2}.$$

Hence Central Limit Theorem.

A "good" uniform approximation $p_n(u) \sim a(u) \cdot B(u)^n$ for $u \approx 1$ (complex neighbourhood) is called QuasiPowers approximation.

From Bender, F.-Soria, Hwang (1995), one has:
— Moments result from differentiation (complex an.)
— Convergence to Gaussian distribution (erf)
— Speed of convergence is $\frac{1}{\sqrt{n}}$.
— Some large deviation estimates: probability of being far from mean at cn for $c \neq \mu$ is exponentially small.

Corollary 2 (Gaussian limit). *For the $\mathcal{T}_{2,3}$ model, the random variable X_n representing the number of balls of the first type at time n is asymptotically Gaussian with speed of convergence to the limit $O(n^{-1/2})$,*

$$\mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}X_n}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. From lattice sum, in complex neighbourhood $u \approx 1$:

$$p_n(u) = K(u)^{-n-1}(1 + O(2^{-n})).$$

Note that $K(u)^{-1}$ plays the rôle of a probability characteristic function but it isn't!

$$K(u)^{-1} \doteq 0.713 + 0.254u^2 + 0.090u^4 - 0.086u^6 + 0.022u^8 + \cdots .$$

The shape of moments.

In the literature, only a few moments are computed via (unpleasant?) recurrence manipulations from probabilities and original rec.

Here: everything is almost as though $p_n(u) = K(u)^{-n-1}$.

$$P_1(\nu) = \frac{4\nu}{7},\ P_2(\nu) = \frac{4\nu}{637}\,(52\,\nu + 17)\,,$$
$$P_3(\nu) = \tfrac{8\nu}{84721}\left(1976\,\nu^2 + 1938\,\nu - 11063\right).$$

Corollary 3 (Moments). *For the $\mathcal{T}_{2,3}$ model, exact polynomial forms for moments of* any *order are available: the factorial moments satisfy*

$$\mathbb{E}((X_n)^{\underline{r}}) = P_r(n+1), \qquad n \geq 6r,$$

where the P_r are polynomials generated by

$$e^{vL(h)} = \sum_{r=0}^{\infty} \frac{h^r}{r!} P_r(v) \quad \textit{and} \quad L(h) = -\log K(1+h).$$

Rota: polynomials of “binomial type” satisfying various convolution relations.

Large deviations.

From dominant poles of ψ, corresponding to $u = 0$:

Corollary 4 (Extreme large deviations). *The probability that, in the $\mathcal{T}_{2,3}$ model, all balls are of the first colour satisfies*

$$[z^{3n+2}]\psi(z) \sim 3\rho^{-3n-3}\left(1 + O(A^{-n})\right),$$

for any $A < 8$.

Moreover:

Corollary 5 (Large deviations). *Let α be a number of the open interval $(0, \frac{4}{7})$. One has*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \leq \alpha \cdot n\right) = -\rho(\alpha), \quad (5)$$

where the rate function ρ is determined by

$$\rho(\alpha) = \log\left(\lambda_0^{\alpha} K(\lambda_0)\right), \quad (6)$$

and λ_0 depending on α is the implicitly defined root $u \in (0, 1)$ of

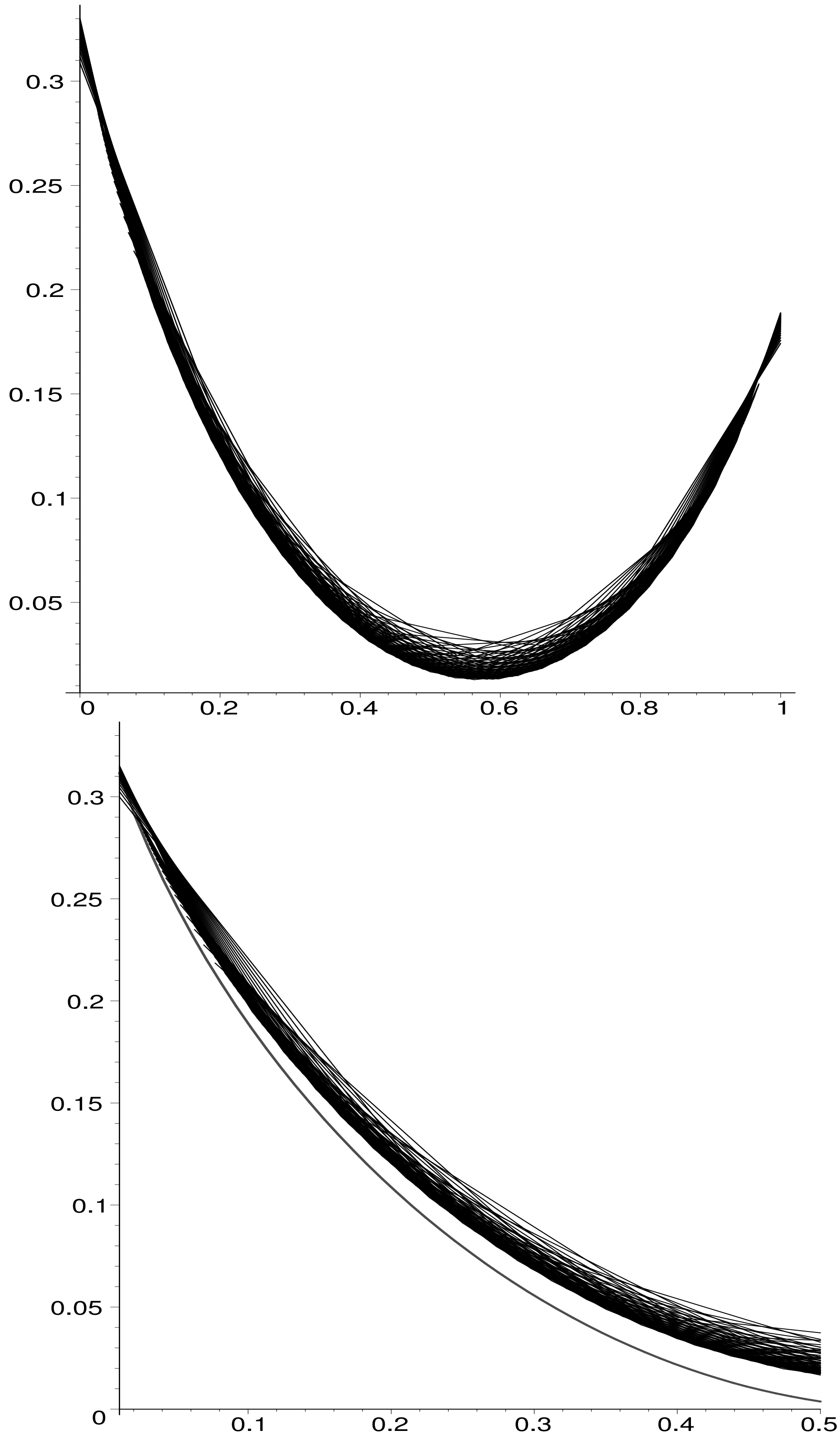
$$\frac{uK'(u)}{K(u)} + \alpha = 0. \quad (7)$$

Proof is standard for probabilists. Assume $p_n(u) \approx B(u)^n$, where $B(u)$ increases from c_0 to 1 as $u \in (0,1)$.

One has Cauchy aka saddle-point bounds:

$$[u^k]p_n(u) \le \frac{p_n(u_0)}{u_0^k} \approx \frac{B(u_0)^n}{u_0^k}.$$

Adopt the best u_0 (which must exist by some convexity prop.) and get an exponentially small upperbound. Cramér aka "shifting the mean": apply a form of CLT near u_0 to conclude that the upperbound is also a lowerbound.



Left: a Sedgewick plot of $\{-\frac{1}{n}\log \mathbb{P}(X_n = k)\}_{k=0}^{n+1}$ for $n = 24\,..\,96$ (the horizontal axis is normalized to $n+1$); right: a comparison against the large deviation rate (thick line).

Related work

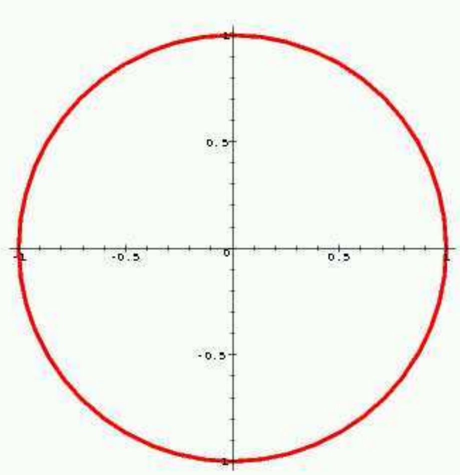
Gaussian law by moments: (Bagchi & Pal 1985). Here global expression for moment polynomials +speed of convergence

Elliptic connection related to (Panholzer & Prodinger 1998) via specific approach $y''' = \cdot\, y'^2 + \cdot$. Here: much more general, for whole class.

Large deviations seem to be new.

Local limit laws? Probably true. Want to apply saddle point, need bounding technique.

Counting	$u = 1$
Moments	$u = 1 \pm \frac{1}{\infty}$
Large deviations	$u = [1-\eta, 1+\eta]$
Central limit	$u = 1 + \square$
Local limit	$\lvert u \rvert = 1$



Part IV

General case

Matrix

drawn	added	
$\downarrow$	B	W
B	α	β
W	γ	δ .

$$\alpha + \beta = \gamma + \delta = s$$

Consider general case of urns with replacement, i.e., $\alpha < 0$, $\beta < 0$.

$$\begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix}$$

A 3–parameter family

Plus initially a_0 black; b_0 white.

Ideas:

• Look at the *enumerative* version.

• Set up PDE for bivariate generating function via operator $e^{z\Gamma}$.

• Get a ψ–function parameterized by Abelian integrals over Fermat curve $x^h + y^h = 1$.

• Determine singularities by looking at geometry of conformal maps of basic domains.

• Generally, non-elliptic solutions, but:
— Gaussian limit with speed of convergence;
— Extreme large deviations;
— Large deviation rate
Recycles most of $\mathcal{T}_{2,3}$ case but without double periodicity at this level of generality.

The operator approach

Number balls in order of appearance $1, 2, 3, \ldots$

$$\overbrace{1_I, 2_I}^{\text{choose 2}}, \quad \overbrace{3_{II}, 4_{II}, 5_{II},}^{\text{choose 5}} \quad \overbrace{6_I, 7_I, 8_I, 9_I,}^{\text{choose 8}} \quad \overbrace{7_I, 9_I, 10_{II}, 11_{II}, 12_{II},}^{\text{choose 9}} \quad \cdot$$

$s = a + b$; at time n, after action, size is t_n;
$t_0 = a_0 + b_0$ is given;

$$t_n = t_0 + sn.$$

Thus

$$H_n = t_0(t_0 + s) \cdots (t_0 + ns).$$

Let $H_{n,k}$ be number of histories of length n leading to k Black (Type I) balls and

$$H(z, u) := \sum_{n,k} H_{n,k} u^k \frac{z^n}{n!}.$$

Combinatorial marking $\equiv$ differentiation

Represent a particular history h with k black balls and ℓ white balls as $u^k v^\ell$.

Evolution chooses a black ball and acts; e.g., for $\mathcal{T}_{2,3}$:

$$u^k v^\ell \underset{u\partial_u}{\Longrightarrow} k u^k v^\ell \underset{u^{-2}v^3}{\Longrightarrow} k u^{k-2} v^{\ell+3}.$$

Similarly for white balls. Cleverly introduce:

$$\Gamma := u^{-1}v^3 \frac{\partial}{\partial u} + u^4 v^{-2} \frac{\partial}{\partial v}.$$

Then $\Gamma u^k v^\ell$ describes all the successors of $h \cong u^k v^\ell$.

All evolutions of length n are generated by $\Gamma^n u^{a_0} v^{b_0}$, and a trivariate version of H is

$$\widehat{H}(z,u,v) := e^{z\Gamma} \circ u^{a_0} v^{b_0}.$$

The basic PDE

- By general principles:

$$\partial_z \left(e^{z\Gamma} f\right) = \Gamma e^{z\Gamma} f, \qquad \partial_z \widehat{H} = \Gamma \circ \widehat{H}.$$

- By homogeneity, any term $\mathfrak{m} = u^\alpha v^\beta z^n$ has $\alpha + \beta = sn + t_0$: $(\theta_u + \theta_v - s\theta_z)\mathfrak{m} = t_0\mathfrak{m},$ where $\theta_u \equiv u\partial_u$.

In summary, system of PDEs:

$$\begin{cases} \partial_z \widehat{H} = \Gamma \circ \widehat{H} \\ (\theta_u + \theta_v - s\theta_z)\widehat{H} = t_0 \widehat{H}. \end{cases}$$

Eliminate ∂_v and get

$$\partial_z \widehat{H} = u^{-a} v^{1+a} \theta_u \widehat{H} + u^{1+b} v^{1-b} \left(s\theta_z \widehat{H} - \theta_u \widehat{H} - t_0 \widehat{H}\right).$$

One can set $v = 1$ and get $H(z, u) = H(z; u, v \mapsto 1)$:

$$\left[(1 - szu^{b+s})\partial_z + (u^{b+s+1} - u^{1-a})\partial_u - t_0 u^{b+s} I\right] \circ H(z, u) = 0.$$

Apply general technology for first-order PDEs.

Theorem 3. *The probability of the urn defined by matrix:* $\begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix}$, *initial cond* : $a_0, t_0 := a_0 + b_0$, *assuming it is tenable, is*

$$p_n(u) = \frac{\Gamma(n+1)\Gamma\left(\frac{t_0}{s}\right)}{s^n \Gamma\left(n+\frac{t_0}{s}\right)}[z^n]H(z,u).$$

There $H(z,u)$ *is given by*

$$H(z,u) = \delta(u)^{t_0}\psi(z\delta(u)^s + I(u)),$$

where $h = s + a + b$,

$$\delta(u) := (1-u^h)^{1/h}, \qquad I(u) := \int_0^u \frac{t^{a-1}}{\delta(t)^{a+b}}\,dt$$

and the function ψ *is defined implicitly by*

$$\psi(I(u)) = \frac{u^{a_0}}{\delta(u)^{t_0}}.$$

Analytic aspects.

Abelian integrals over Fermat curve

$$x^h + y^h = 1.$$

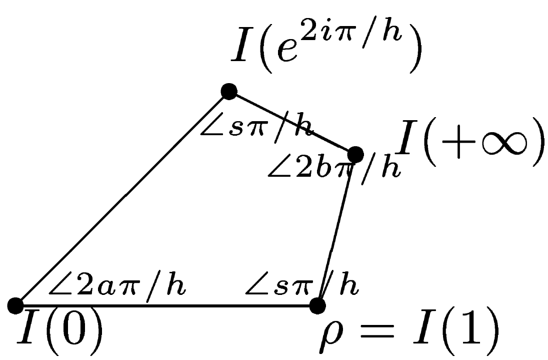
In general, global structure is not "clear", but dominant singularities are OKay.

Consider the complex plane with h rays emanating from 0 and having directions given by all the hth roots of unity.

$$S_j := \left\{ z, \quad z = Re^{i\theta},\ 0 < R < \infty,\ \frac{2j\pi}{h} < \theta < \frac{2(j+1)\pi}{h} \right\}.$$

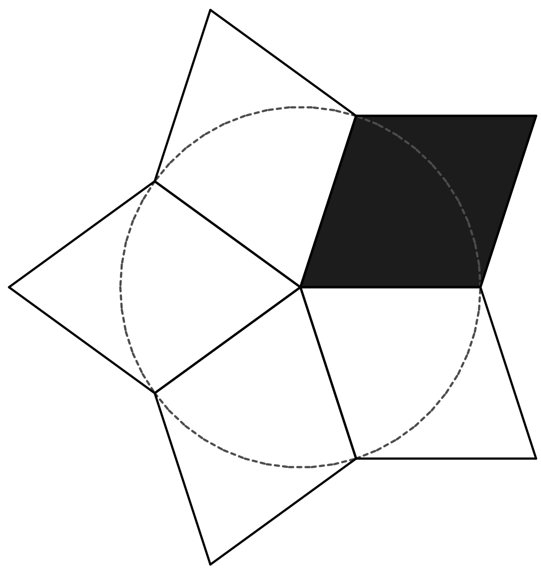
The image of S_0 by $I(u)$ is a quadrilateral, the elementary kite with vertices at the points

$$0, \quad I(1), \quad I(+\infty), I(e^{2i\pi/h}).$$



The elementary kite.

Definition 1. *The* fundamental polygon *of an urn model is the (closure of) the union of h regularly rotated versions of the elementary kite about the origin.*



The elementary kite and the fundamental polygon of the urn

$$\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$$

$h = 5$, $s = 3$

$$\psi(z) \asymp (\rho - z)^{-1/3}.$$

Theorem 4. *The ψ function is analytic beyond its disc of convergence whose radius is*

$$\rho = \frac{1}{h} B(\frac{a}{h}, \frac{s}{h}) = \frac{1}{h} \frac{\Gamma(\frac{a}{h})\Gamma(\frac{s}{h})}{\Gamma(\frac{a+s}{h})}.$$

It has an algebraic branch point at $z = \rho$, where

$$\psi(\rho - x) \asymp (\rho - x)^{-a/s}. \tag{8}$$

It is continuable beyond its circle of convergenc in a star-like domain.

Proof. Uses symmetries about origin, then rotations around vertices.

Suffices to apply singularity analysis.

Probabilisitic consequences

Corollary 6. *A quasipowers approximation holds but with weaker error terms than* $\mathcal{T}_{2,3}$. *The limit law is* Gaussian *with speed of convergence* $O(\frac{1}{\sqrt{n}})$.

Corollary 7. *The large deviation rate exists and is expressible in terms of integrals over the Fermat curve.*

Corollary 8. *The extreme large deviation rate is given explicitly in terms of Gamma function values at rational points.*

Part V

Special cases and explicitly solvable models

The urns

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

correspond to: sampling with replacement or without replacement, and Coupon Collector.

Solutions agree with basic combinatorics!

$$H(z,u) = u^{a_0} e^{(a_0+b_0)z}$$

$$H(z,u) = (z+u)^{a_0}(z+1)^{b_0}.$$

$$H(z,u) = (e^z - 1 + u)^{a_0}.$$

The Ehrenfest Urn • Initially: 2 urns with balls moving between them.

• A celebrated controversy: *irreversibility versus ergodicity*.

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Balance is $s = 0$. One has $a = b = 1$, hence $h = 2$. Start with $a_0 = m$.

One has $\delta(u) = (1 - u^2)^{1/2}$, hence genus 0.

$$I(u) = \int_0^u \frac{dt}{1 - t^2} = \frac{1}{2} \log \frac{1+u}{1-u} = \operatorname{atanh}(u).$$

The function ψ is defined implicitly by

$$\psi(\operatorname{atanh}(u)) = \left(\frac{u}{\sqrt{1-u^2}} \right)^m,$$

which is equivalent to $\psi(w) = \sinh^m w$.

$H(z, u) = (1-u^2)^{m/2} \sinh^m(z + \operatorname{atanh} u) = (\sinh z + u \cosh z)^m.$

$\equiv$ **Combinatorics!**

Elliptic cases

$$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Corollary 9. *The three urn models A, B, C of balance 1 have solutions expressible in terms of elliptic functions. The corresponding lattices are the equilateral triangular lattice (cases A, C) and the square lattice tilted by $\pi/4$ (case B).*

Like for $\mathcal{T}_{2,3}$: **TILINGS**.

$$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$

Corollary 10. *The urn model D admits an elliptic function solution of the lemniscatic type.*

Urns without replacement

= the original models!

♡ Pólya–Eggenberger's contagion urn.

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

$$H(z,u) = \frac{u^{a_0}}{(1-az)^{b_0/a}(1-au^a z)^{a_0/a}},$$

With $a = 1$ and $a_0 = b_0 = 1$, the PGF of at time n is

$$\frac{u}{n+1}(1+u+\cdots+u^n),$$

Cf. also M. Durand. In general:

$$\mathbb{P}(\text{White}_n = a_0 + ja) = \frac{[z^n u^j]\,(1-z)^{-b_0/a}(1-uz)^{-a_0/a}}{[z^n]\,(1-z)^{-t_0/a}}.$$

The altruistic model.

$$T = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Friedman 1947: "Every time an accident occurs, the safety campaign is pushed harder. Whenever no accident occurs, the safety capaign slackens and the probability of an accident increases."

$$H(z,u) = u^{a_0} e^{a_0 z(1-u^a)} \frac{(1-u^a)^{t_0/a}}{(1-ue^{az(1-u^a)})^{t_0/a}},$$

Smythe $a=1$: stemma construction in philology as well as with recursive trees. Eulerian numbers and leaves in "recursive trees".

The KMR urn: Kotz–Mahmoud–Robert!

$$\begin{pmatrix} a+1 & 0 \\ 1 & a \end{pmatrix},$$

Bagchi and Pal (1985): “present some curious technical problems”.

Bivariate algebraic solution, genus 0:

$$u^{t_0}\left(1-\frac{1-u^a}{(1-(a+1)u^{a+1}z)^{a/(a+1)}}\right)^{(a_0-t_0)/a}\left(1-(a+1)u^{a+1}z\right)^{-t_0}$$

Mean and variance at time n ($a=3$):

$$4n+1-\frac{1}{\binom{n-3/4}{n}}\sim 4n-\frac{\pi\sqrt{2}}{\Gamma(3/4)}n^{3/4}+1+O(n^{-1/4}).$$

$$\frac{2}{3}\frac{8\sqrt{2}-3\pi}{\Gamma(3/4)^2}n^{3/2}-\frac{3\pi\sqrt{2}}{\Gamma(3/4)}n^{3/4}+O(\sqrt{n}).$$

Distribution: prototype is

$$\widehat{H}(z,u)=\left(1-u(1-(1-z)^{a/(a+1)})\right)^{-1/a}$$

Singular exponent is discontinuous at $u=1$.

Banderier, F., Schaeffer, Soria: within analytic combinatorics such changes are associated to stable laws. (Modify singularity analysis techniques.)
— e.g., cores of random maps.

Corollary 11. *Model with matrix* $(a+1, 0, 1, a)$ *and* $t_0 = 1, a_0 = 0$*:*

$$\mathbb{P}\left(\frac{X_n}{n^{a/(a+1)}} = x\right) \sim \frac{1}{n^{a/(a+1)}} \frac{\Gamma(1/(a+1))}{\Gamma(1/a)} x^{1/a-1} G\left(x; \frac{a}{a+1}\right),$$

$$G(x;\lambda) = -\frac{1}{\pi} \sum_{j\geq 1} \frac{(-x)^j}{j!} \Gamma(1+\lambda k) \sin(k\pi\lambda),$$

the quantity $x^{-1}G(x^{-\lambda};\lambda)$ *is exactly the density of a stable law of index* λ *when* $0 < \lambda < 1$.

Supplements *martingale arguments* of Gouet (1993) = nonconstructive.

Conclusions

A unified analytic framework

All 2×2 urns with constant balance admit of analytic model.

Some interesting special function solutions: algebraic, elliptic, etc.

Some new probability laws.

Work still in progress!