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Counting Domino Tilings of Rectangles via Resultants

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Abstract

The classical cosine formula for enumerating domino tilings of a rectangle, due to Kasteleyn, Temperley, and Fisher is proved using a combination of standard tools from combinatorics and algebra. For further details see [4].

1. Introduction

A classical result in combinatorial enumeration, first proved by Kasteleyn [3] gives the number of domino tilings of an $m \times n$ rectangle (mn even) as

$$k_{m,n} = \prod_{j=1}^{\lceil m/2 \rceil} \frac{c_j^{n+1} - d_j^{n+1}}{2b_j}$$

with $b_j = \sqrt{1 + \cos^2 \frac{j\pi}{m+1}}$, $c_j = b_j + \cos \frac{j\pi}{m+1}$, and $d_j = b_j - \cos \frac{j\pi}{2m+1}$.

The result can be written in a nicer way when m and n are even to get the "cosine formula:"

$$k_{2m,2n} = 4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right)$$

Here is a new proof of this cosine formula. It uses the following notions:

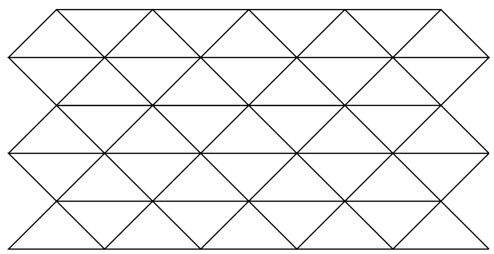
- the method of determinant evaluation by counting families of non-intersecting paths in a graph,
- the inversion formula relating heaps and trivial heaps in a commutation monoid,
- in the particular case of a line, the interpretation of heaps in terms of lattice paths and their relation to the matching polynomials ,
- the determinant evaluations due to Laplace and Binet–Cauchy
- the Sylvester matrix of two polynomials and its determinant, the resultant.

These notions are explained in Section 2 of the full paper [4]. We now concentrate on the proof. The idea is to show that the number of domino tilings of a $(2m \times 2n)$ rectangle can be expressed as a resultant of two matching polynomials from which the cosine formula can be deduced. In Section 3 a multivariate version is given.

2. The Proof

2.1. **From tilings to paths.** Domino tilings of a $2m \times 2n$ rectangle can be coded by systems of vertex-disjoint paths in a particular graph which is part of the Generalized Pascal Triangle. The graph $\Gamma_{m,n}$ can be defined as a graph whose vertices are the lattice points $(i,j) \in \mathbb{Z}$ for $0 \le i \le 2n$, $0 \le j \le 2m$, and $i+j$ even, and whose vertex (i,j) has three outgoing edges to vertices $(i+1,j+1)$, $(i+2,j)$, and $(i+1,j-1)$.

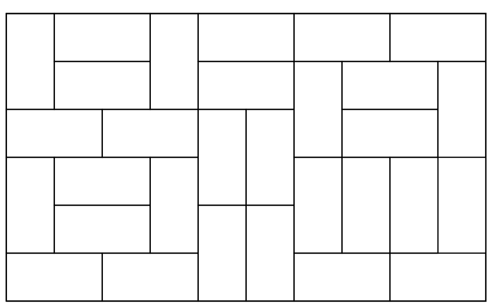
The m sources are the points of abscissa 0 and the m targets are the points of abscissa $2n$. The ith source has coordinates $\big(0, 2(i-1)\big)$ and the ith target has coordinates $\big(2n, 2(i-1)\big)$. An example of the graph $\Gamma_{3,4}$ is given below:

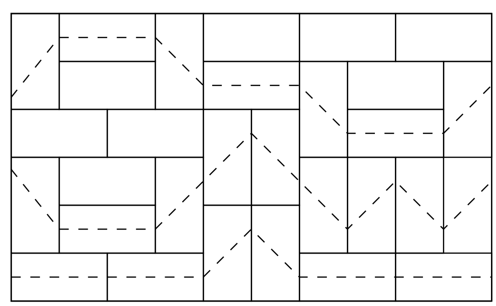


Domino tilings are in bijection with sets of m non-intersecting paths on $\Gamma_{m,n}$. Given a tiling, start on the left side and traverse the tiled rectangle according to the rules:

- if a vertical tile is hit traverse diagonally,
- if a horizontal tile is hit traverse straight.

Starting with a tiling on the 6×8 rectangle an example of the bijection is illustrated:





Using the theory of non-intersecting paths [1], this shows that $k_{2m,2n} = \det H_{m,n}$ where the entry $h_{i,j}$ in $H_{m,n}$ is the number of paths from the ith source to the jth target.

2.2. **Extending the graphs of the path.** Now $\Gamma_{m,n}$ is extended to the left and to the right to create a new graph $\bar{\Gamma}_{m,n}$ by adding to it:

- vertices $(i,j) \in \mathbb{Z}$ for $2n < i < 2n+2m$, $2n-2m < i-j < 2n$, and $i+j$ even,
- vertices $(i,j) \in \mathbb{Z}$ for $-2m+2 \le i < 0$, $-2m+2 \le i-j < 0$, and $i+j$ even,

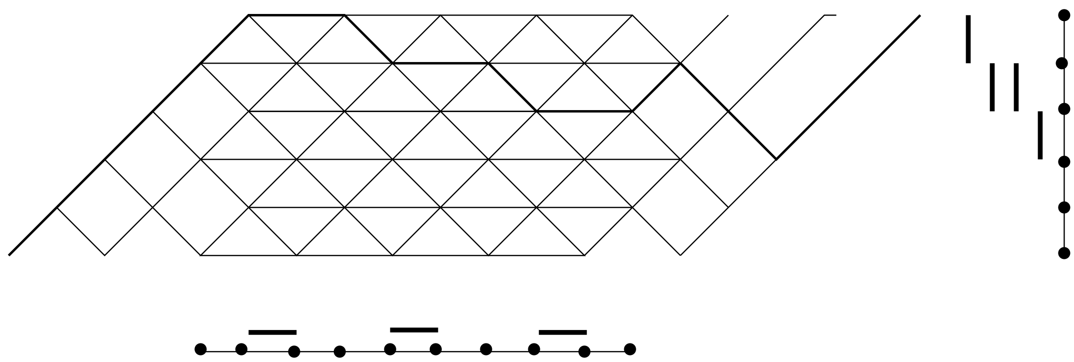
and by connecting among themselves the added vertices and the vertices of $\Gamma_{m,n}$ whenever NE-edges and SE-edges are possible.

An example of the graph $\bar{\Gamma}_{3,4}$ is given in Section 2.3.

In that graph the ith source has coordinates $(-2i+2, 0)$ and the jth target $(2n+2m-2j+1, 2m-2)$. It is obvious that the number of systems of vertex-disjoint paths on $\Gamma_{m,n}$ is equal to the number of systems of vertex-disjoint paths on $\bar{\Gamma}_{m,n}$. This shows that $k_{2m,2n} = \det \bar{H}_{m,n}$ where the entry $\bar{h}_{i,j}$ in $\bar{H}_{m,n}$ is the number of paths from the ith source to the jth target on $\bar{\Gamma}_{m,n}$.

2.3. **Splitting the paths.** Let $\mathcal{L}_n$ denote the graph of $(n+1)$ points on a line. Given a path leading from the ith source of $\bar\Gamma_{m,n}$ to the jth target, the horizontal steps define a trivial heap of $\mathcal{L}_{2n-1}$ and the up-down steps are equivalent to a heap of $\mathcal{L}_{2m-1}$.

An example is given below:



If the path has k horizontal steps, then the trivial heap has k pieces and the resulting heap has $n+i-j-k$ pieces. Let $f_{n,k}$ (resp. $g_{m,k}$) be the number of trivial heaps (resp. heaps) with k pieces on $\mathcal{L}_{2n-1}$ (resp. $\mathcal{L}_{2m-1}$). Then we define $m\times(m+n)$ matrices

$$F_{m,n} = [f_{n,i-j}]_{0\le i<m, 0\le j<m+n} \text{ and } G_{m,n} = [g_{m,n+i-j}]_{0\le i<m,0\le j<m+n}.$$

Then $\bar H^t_{m,n} = F_{m,n} G^t_{m,n}$.

2.4. **Dualizing path systems.** According to the Binet–Cauchy formula

$$\det F_{m,n}G^t_{m,n} = \sum_{J\in\binom{[m+n]}{n}} \det F_{m,n}\langle J\rangle \det G^t_{m,n}\langle J\rangle.$$

Let $\Phi_{m,n}$ be the graph consisting of $m+n$ horizontal lines joined by vertical edges labeled from 1 to $2n-1$ as follows for $n=3$ and $m=4$. It has $m+n$ sources $\mathbf{u}(u_1,\dots,u_{m+n})$ and $m+n$ targets $\mathbf{v}=(v_1,\dots,v_{m+n})$. The vertical edges are directed from top to bottom. The Gessel–Viennot machinery [1, 2] says that:

- $\det F_{m,n}\langle J\rangle$ = non-intersecting paths in $\Phi_{m,n}$ from $u_{[m]}$ to v_J,
- $\det G^t_{m,n}\langle J\rangle$ = non-intersecting paths $\Phi_{n,m}$ from $u_{[n]}$ to $v_{[n+m]\setminus J}$.

Therefore

$$\det G^t_{m,n}\langle J\rangle = \det F_{m,n}\langle [m+n]\setminus J\rangle.$$

2.5. **The resultant appears.** Having

$$\det F_{m,n}G^t_{m,n}\langle J\rangle = \sum_{J\in\binom{[m+n]}{m}} \det F_{m,n}\langle J\rangle \det F_{m,n}\langle [m+n]\setminus J\rangle = \det\begin{bmatrix} F_{m,n}\\ F'_{m,n}\end{bmatrix}$$

with $F'_{m,n}$ is the matrix $F_{m,n}$ where all the elements are multiplied by $(-1)^{m+n}$.

Now we have a Sylvester matrix and

$$\det\begin{bmatrix} F_{m,n}\\ F'_{m,n}\end{bmatrix} = \text{resultant}\big(f_n(t), f_m(-t)\big)$$

with

$$f_0(t)=1,\ f_1(t)=1+t,\ f_{n+1}(t)=(t+2)f_n(t)-f_{n+1}(t).$$

2.6. **The formula.** Now to get the formula, $f_n(t)$ can be written as:

$$f_n(t) = \prod_{j=1}^{n} \left(t + 4\cos^2 \frac{j\pi}{2n+1} \right)$$

and for two monomial polynomials $a(t)$ and $b(t)$ with roots α_i, $1 \le i \le n$ and β_j, $1 \le j \le m$:

$$\text{resultant}_t(a,b) = a_0^n b_0^m \prod_{i=1}^{n} \prod_{j=1}^{n} (\alpha_i - \beta_j).$$

The cosine formula formula follows directly.

3. A Multivariate Refinement

The counting can be refined. To each tiling one can associate a monomial $c_t(x,y)$ in the variables $\mathbf{x} = (x_1, \ldots, x_{2n-1})$ and $\mathbf{y} = (y_1, \ldots, y_{2m-1})$. The information about the positions of horizontal and vertical tiles can be carried over the path systems in the graph $\Gamma_{m,n}$. The edges will get a weight as follows:

– an horizontal edge $(i,j) \to (i+2,j)$ gets weight x_{i+1}.
– an up-edge $(i,j) \to (i+2,j)$ gets weight 1.
– an down-edge $(i,j) \to (i+1,j-1)$ gets weight y_j.

Then generalized matching polynomials $f_n(\mathbf{x};t) = f_n(x_1,\ldots,x_{2n-1};t)$ are introduced:

$$f_0(-;t) = 1;\ \ f_1(x_1;t) = t + x_1;\ \ f_{n+1} = (t + x_{2n} + x_{2n+1}) f_n(\mathbf{x};t) + x_{2n}x_{2n-1} f_{n-1}(\mathbf{x};t).$$

It is easy to check that the proof of Section 2 goes through.

$$k_{2m,2n}(\mathbf{x},\mathbf{y}) = \text{resultant}\big(f_n(\mathbf{x};t), f_m(\mathbf{y};t)\big)$$

This can be also interpreted in terms of 2-tableaux [4].

If we set $x_i = x$ and $y_i = y$, the cosine formula counting horizontal and vertical tiles separately [3]:

$$k_{2m,2n} = 4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(y\cos^2 \frac{j\pi}{2m+1} + x\cos^2 \frac{k\pi}{2n+1} \right)$$

Now to consider the tiling of an $2m \times (2n-1)$ rectangle it suffices to set up the counting machinery for a $2m \times 2n$ rectangle and to set $x_{2n-1} = 0$ in order to have the last column of the rectangle covered with vertically oriented dominos. Then in the resultant the polyonial $f_n(t)$ has to be replaced by $\tilde{f}_n(t) = f_n(t) - f_{n-1}(t)$.

If both side lengths are odd, the same idea applies, but the polynomials always have t as a factor. This implies that the resultant vanishes which algebraically reflects the obvious combinatorial fact that a rectangle with an odd area can not be tiled by dominos.

Some other specializations can be find in the full paper [4].

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