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Asymptotic Analysis of TCP Performances Under Mean-field Approximation

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Abstract

The talk deals with the performances of TCP when N connections share the same router with high capacity NT. Using mean field approximation, some asymptotic results on the throughput and the distribution of the size of the congestion window when N is large are established. The talk is based on a joint paper with Cédric Adjih and Nikita Vvedenskaya [1].

1. The Real Protocol and the Models

TCP (Transmission Control Protocol) is the protocol which controls 99.9% of the traffic on the Internet network. It is a end-to-end protocol where the destination sends acknowledgments to the source, which controls its congestion window according to them and retransmits the lost packets.

We present the study of the multi-connection case where N connections share the same router with finite capacity. Every user is loading a file with infinite size via a unique connection and adapts its congestion window according to TCP, which can be roughly described as follows: the size of the congestion window is increased by one each time a number of acknowledgments equal to the window size has been received; each time there is a packet loss, the user halves the size of its congestion window. Losses are due to the finite capacity of the buffer which receives packets from all the users. The model under study does not take into account refinements of the protocol like the *slotted time*, the *slow start*, and the *self-clocking* (see [1] for details).

The time between sending a packet and receiving the acknowledgment is called round-trip time (RTT). In the buffer with capacity TN, the service time of a packet is one and the RTT has an exponential distribution with mean N/λ. This model is highly unrealistic but analytically tractable. Nevertheless the analysis can be generalized to a RTT that is the sum of a fixed term NT and a delay with an exponential distribution with mean N/λ, much smaller than NT. This second model is much more realistic.

2. The Asymptotic Case

Let $R^N(t)$ be the free capacity in the buffer at time t and $W_i^N(t)$ the size of the congestion window of user i at time t. Let $R^N(x,t) = \mathbf{P}(R^N(t) > x)$ and $w^N(y,t) = -\frac{\partial}{\partial y} W^N(y,t)$ the density of the window size distribution. According to the dynamic of the system, the following equations hold:

$$\frac{\partial}{\partial t}R^N(x,t) = -\frac{\partial}{\partial x}R^N(x,t) + \frac{\lambda}{N}\sum_{i=1}^{N}\Big(R^N(x+W_i^N(t),t) - R^N(x,t)\Big), \tag{1}$$

$$\frac{\partial}{\partial t}w^N(y,t) = \frac{\lambda}{N}\Big(R^N(y-1,t)w^N(y-1,t) + (1-R^N(2y,t))w^N(2y,t) - w^N(y,t)\Big). \tag{2}$$

These equations show a separation of time scales: $R^N(t)$ varies at rate of order λ and $w^N(y,t)$ obviously varies at rate λ/N. Therefore when N is large, $w^N(y,t)$ tends to be slowly varying and $R^N(t)$ reaches its steady state distribution $\tilde{R}$ where $w(y,t)$ is independent of t. When N is large, $R^N(t)$ converges to $R(t)$ satisfying, using Equation (1),

$$\frac{\partial}{\partial t}R(x,t) = -\frac{\partial}{\partial x}R(x,t) + \left(\int_0^{+\infty} R(x+y,t)\,dW(y) - R(x,t)\right)\lambda.$$

Thus $R(t)$ has a stationary limit $\tilde{R}$ which has an exponential distribution with parameter $a > 0$ such that $\lambda\Big(1-\mathbf{E}\big(\exp(-aW)\big)\Big) = a$. Heuristically, when t tends to infinity, if $a(t)$ tends to a limit a then the limit solution $w(y)$ is the solution of

$$w(y) = e^{-a(y-1)}w(y-1) + (1-e^{-2ay})w(2y) \tag{3}$$

where

$$\left(1 - \int_0^{+\infty} e^{-ay}w(y)dy\right)/a = 1/\lambda.$$

In the case $a \ll 1$ (i.e., the loss rate tends to 0) and with the approximation that $w(y) = \sqrt{a}g(y\sqrt{a}) + O(a)$, Equation (3) becomes at first order

$$yg(y) + g'(y) = 2yg(2y). \tag{4}$$

It can be solved introducing its Mellin transform $g^*(s) = \int_0^{+\infty} g(y)y^{s-1}\,dy$ and it comes that

$$g(y) = \frac{2}{\pi}\prod_{k\geq 1}(1-4^{-k})^{-1}\sum_{n\geq 0} a_n 2^n \exp(-4^n y^2/2)$$

where $\sum_{n\geq 0} a_n x^n = \prod_{k\geq 1}(1-4^{-k}x)$. This result can be compared to the result by Dumas et al. [2]. Is is proved in their paper that if $\tilde{W}$ is the congestion window size just before a loss then $\tilde{W}^2$ has the same distribution as $2\sum_{k\geq 1} 2^{-2^k} I_k$ where I_k are i.i.d. variables with exponential distribution of parameter 1.

These analytical results, typically the distribution of the size of the congestion window, have been compared with simulations of two types: the previous *simplified* model of TCP and the *real* TCP, using the simulator ns2. In both cases of simulations, an oscillation of the size of the buffer occupancy from the limit capacity has been observed. The analytical mean value of the free buffer size agrees with the simulation of *simplified* TCP.

Bibliography

[1] Adjih (Cédric), Jacquet (Philippe), and Vvedenskaya (Nikita). – *Performance evaluation of a single queue under multi-user TCP/IP connections*. – Research Report n° 4141, Institut National de Recherche en Informatique et en Automatique, March 2001. 40 pages.

[2] Dumas (Vincent), Guillemin (Fabrice), and Robert (Philippe). – A Markovian analysis of additive-increase multiplicative-decrease algorithms. *Advances in Applied Probability*, vol. 34, n° 1, 2002, pp. 85–111.