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Multi-Variable sinc Integrals and the Volumes of Polyhedra

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Abstract

This talk investigates integrals of the form

$$\tau_n := \int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx$$

and their multi-dimensional analogues. These integrals are related to volumes of polyhedra, which allows to derive various monotony results of such integrals.

1. Introduction and Motivation

A conjecture stated that

$$\mu := \int_0^\infty \prod_{k=1}^{\infty} \cos\left(\frac{x}{k}\right)\, dx < \frac{\pi}{4}. \tag{1}$$

Indeed, $\mu \approx 0.7853\underline{80}$, while $\frac{\pi}{4} \approx 0.7853\underline{98}$ differs in the fifth place. The highly oscillatory integral of an infinite product of cosines (1) is connected to the integrals

$$\tau_n := \int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx,$$

where $\operatorname{sinc}(\cdot)$ is the *sine cardinal* function,[1] defined by

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Section 2 investigates the behavior of the integrals τ_n as functions of n and exhibits a duality between the τ_n and volume of polyhedra. This duality allows to derive various monotony results for the τ_n and to extend the one-dimensional analysis to the multi-dimensional case, which is sketched in Section 3. Section 4 returns to the integral μ and proves Conjecture (1). Some material contained in this summary is taken from [2].

[1] See, e.g., `http://mathworld.wolfram.com/SincFunction.html`.

2. **Fourier Transform and** sinc **Integrals**

2.1. **Fourier cosine transform.** This section recalls some standard results about the Fourier cosine transform (FCT) [3, §13].

Definition 1. The FCT of a function $f \in \mathcal{L}^2(-\infty, \infty)$ is defined to be the $\mathcal{L}^2$-limit $\hat{f}$, if it exists, as $y \to \infty$ of the functions

$$c_y(x) := \frac{1}{\sqrt{2\pi}} \int_{-y}^{y} f(t) \cos(xt)\, dt.$$

Property 1. *The function $\hat{f}$ exists, belongs to $\mathcal{L}^2$ and is unique, apart from sets of zero Lebesgue measure.*

Property 2. *If f is continuous over $(-\alpha, \alpha)$ for some $\alpha > 0$ and if $\hat{f} \in \mathcal{L}^1(-\infty, \infty)$ then, conversely, for $t \in (-\alpha, \alpha)$*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) \cos(xt)\, dt = f(t).$$

Property 3 (Convolution)**.** *If $\hat{f}_1$ and $\hat{f}_2$ are the FCTs of* even *functions f_1 and f_2 in $\mathcal{L}^2(-\infty, \infty)$, then $\hat{f}_1 \hat{f}_2$ is the FCT of $\frac{1}{\sqrt{2\pi}} f_1 * f_2$, where $*$ denotes the convolution product over $(-\infty, \infty)$.*

Property 4 (Parseval)**.** *With the same notations as in Property 3 and provided that at least one of the functions f_1 or f_2 is real, then*

$$\int_0^{\infty} f_1(t) f_2(t)\, dt = \int_0^{\infty} \hat{f}_1(x) \hat{f}_2(x)\, dx.$$

The function χ_a, for $a > 0$, is defined by

$$\chi_a(x) := \begin{cases} 1 & \text{if } |x| < a \\ \frac{1}{2} & \text{if } |x| = a \\ 0 & \text{if } |x| > a. \end{cases}$$

The FCT of χ_a is $a\sqrt{\frac{2}{\pi}}\,\mathrm{sinc}(ax)$ and, conversely, the FCT of $a\sqrt{\frac{2}{\pi}}\,\mathrm{sinc}(ax)$ is *equivalent* to χ_a. Note that the functions χ_a and sinc are both even and real functions and they both belong to $\mathcal{L}^1(0, \infty) \bigcap \mathcal{L}^2(0, \infty)$, which fulfills the hypotheses of the above properties.

2.2. **Duality.** One first introduces the following notations

$$\sigma_n := \prod_{k=1}^{n} \mathrm{sinc}(a_k x), \qquad s_n := \sum_{k=1}^{n} a_k,$$

$$f_n := \frac{1}{a_n}\sqrt{\frac{\pi}{2}}\chi_{a_n}, \qquad F_0 := f_0, \qquad F_n := \left(\sqrt{2\pi}\right)^{1-n} f_1 * f_2 * \cdots * f_n, \text{ for } n \geq 1.$$

By Property 3, one gets that F_n is the FCT of σ_n, and that σ_n is the FCT of F_n. Now, applying Property 4 leads to

$$\tau_n = \int_0^{\infty} F_0(x) F_n(x)\, dx \underset{\text{convolution}}{=} \frac{1}{a_0}\sqrt{\frac{\pi}{2}} \int_0^{\min(s_n, a_0)} F_n(x)\, dx, \tag{2}$$

provided that $\tau_0 = \pi(2a_0)^{-1}$, which is a standard result [1, p. 314].

Consider the hyper-cube H_n and the polyhedron P_n defined by

$$H_n := \{\, (x_1, \dots, x_n) \;\big|\; |x_k| \le 1,\ k \in [\,1, n\,] \,\},$$

$$P_n := \left\{ (x_1, \dots, x_n) \;\middle|\; \left| \sum_{k=1}^{n} a_k x_k \right| \le a_0,\ |x_k| \le 1, k \in [\,1, n\,] \right\},$$

then (2) reads

$$\tau_n = \frac{\pi}{a_0} \frac{1}{2^n a_1 \dots a_n} \int_0^{\min(s_n, a_0)} \chi_{a_1}(x) * \cdots * \chi_{a_n}(x)\, dx = \frac{\pi}{2a_0} \frac{\mathrm{Vol}(P_n)}{\mathrm{Vol}(H_n)}, \tag{3}$$

where $\mathrm{Vol}(\cdot)$ denotes the volume. Equation (3) expresses a *duality* between the integrals τ_n and the volumes of polyhedra. This duality is used to prove the following theorem.

Theorem 1 (Monotony)**.** *For $a_k \ge 0$, then*

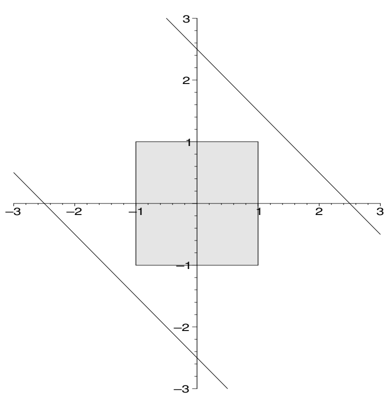
$$0 < \tau_n \le \frac{1}{a_0} \frac{\pi}{2} \qquad \textit{with equality if } a_0 \ge s_n,$$

$$0 < \tau_{n+1} \le \tau_n < \frac{1}{a_0} \frac{\pi}{2} \qquad \textit{provided that } a_{n+1} \le a_0 < s_n.$$

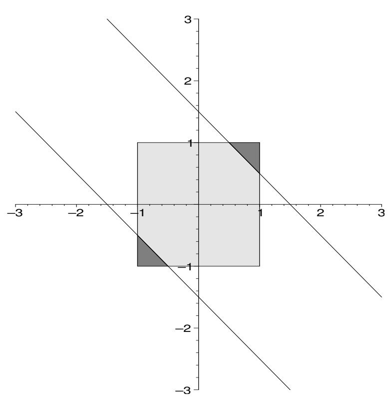
2.3. **Some puzzling integrals.** Consider the family τ_n, where $a_k = \frac{1}{2k+1}$. For $k \in [\,0, 6\,]$, $\tau_k = \frac{\pi}{2}$. However,

$$\tau_7 = \frac{467807924713440738696537864469}{935615849440640907310521750000} \pi \approx 0.499999999992646\pi.$$

According to Theorem 1, this result is explained by the fact that the value of the integrals τ_n drops when the constraint $\sum_{k=1}^{n} a_k x_k \le a_0$ *bites* into the hyper-cube H_n. Indeed, $\sum_{k=1}^{6} a_k < 1$, but on the addition of the seventh term, the sum exceeds 1 and the identity $\tau_k = \frac{\pi}{2}$ no longer holds. This behavior is illustrated in the case of dimension 2 by the following diagrams.



Volume = □



Volume = □ − 2△

3. **Multi-Dimensional** sinc **Integrals**

Let $a := (a_1, \dots, a_m)$ and $y := (y_1, \dots, y_m)$ in $\mathbb{R}^m$. Define $ay := \sum_{k=1}^{m} a_k y_k$ and δ_a the Lebesgue measure restricted to $\{\, x \in \mathbb{R}^m \mid x = ta,\ -1 \le t \le 1 \,\}$. For any integrable function f over $\mathbb{R}^m$, $\int_{\mathbb{R}^m} f(x) \delta_a(dx) = \int_{-1}^{1} f(ta)\, dt$ and thus

$$\int_{\mathbb{R}^m} e^{ixy} \delta_a(dx) = 2 \operatorname{sinc}(ay). \tag{4}$$

More generally, with $s_1, \ldots, s_n \in \mathbb{R}^m$ and the convolution measure $\lambda = \delta_{s_1} * \cdots * \delta_{s_n}$, Equation (4) becomes

$$F(y) := \int_{\mathbb{R}^m} e^{ixy} \lambda(dx) = 2^n \prod_{k=1}^{n} \operatorname{sinc}(s_k y).$$

Another version of Parseval's theorem yields the following theorem.

Theorem 2. *With the same notations as above and with $n \geq m$ and the constraint that the $m \times m$ matrix $(s_1, \ldots, s_m)$ is non-singular, then*

$$\int_{\mathbb{R}^m} F(y) \prod_{k=1}^{m} \operatorname{sinc}(y_k)\, dy = \frac{\pi^m}{2^n} \int_{[\,-1,1\,]^m} \lambda(dy).$$

This theorem relates the volume of a polyhedra of dimension n with a m-dimensional sinc integral.

4. The Cosine Integrals Revisited

Invoking the factor theorem of Weierstrass [4, p. 137], one gets

$$\operatorname{sinc}(x) = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right) \text{ and } \cos(x) = \prod_{l=0}^{\infty} \left(1 - \frac{4x^2}{\pi^2 (2l+1)^2}\right).$$

If one lets $C(x) = \prod_{k=1}^{\infty} \cos\left(\frac{x}{n}\right)$, it follows that $C(x) = \prod_{k=0}^{\infty} \operatorname{sinc}\left(\frac{2x}{2k+1}\right)$. By Theorem 1, where $a_k = \frac{2}{2k-1}$, one obtains

$$0 < \mu = \int_0^{\infty} C(x)\, dx = \lim_{n \to \infty} \int_0^{\infty} \prod_{k=1}^{n} \operatorname{sinc}(a_k x)\, dx < \frac{\pi}{4},$$

which proves the conjecture stated in Equation (1).

Bibliography

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[4] Whittaker (E. T.) and Whatson (G. N.). – *A course of modern analysis.* – Cambridge University Press, 1927, fourth edition.