Algorithms Seminar 2000–2001,
F. Chyzak (ed.), INRIA, (2002), pp. 31–36.

Available online at the URL
http://algo.inria.fr/seminars/.

Eulerian Calculus: a Technology for Computer Algebra and Combinatorics

Dominique Foata
Département de mathématique, Université Louis Pasteur (France)

May 21, 2001

Summary by Dominique Gouyou-Beauchamps

Abstract

Babson and Steingrímsson have introduced pairs of permutation statistics that they conjectured were all Euler–Mahonian, i.e., equidistributed with the pair $(\mathrm{des}, \mathrm{maj})$ where des is the number of descents and maj is the major index. How to prove their conjecture? We use the so-called "Umbral Transfer Matrix Method" implemented by Zeilberger and specific combinatorial constructions leading to new transformations on the symmetric group. Details may be found in the recent work of D. Foata and D. Zeilberger [2].

1. Introduction

We use the Babson–Steingrímsson notation [1] for "atomic" permutation statistics. Given a permutation $w = x_1 x_2 \dots x_n$ of 1, 2, …, n they denote $(a - bc)(w)$ the number of occurences of the pattern $a - bc$, i.e., the number of pairs of places $1 \le i < j < n$ such that $x_i < x_j < x_{j+1}$. Similary, the pattern $(b - ca)(w)$ is the number of ocurrences of $x_{j+1} < x_i < x_j$, and in general, for any permutation α, β, γ of a, b, c, the expression $(\alpha - \beta\gamma)(w)$ is the number of pairs (i, j), $1 \le i < j < n$, such that the orderings of the two triples (x_i, x_j, x_{j+1}) and α, β, γ are identical. The statistic $(ab - c)$ is defined in the same way by looking at the occurences (x_i, x_{i+1}, x_j) such that $i + 1 < j$ and $x_i < x_{i+1} < x_j$. Of course, $(ba)(w)$ denotes the number $\mathrm{des}\, w$ of *descents* of w (i.e., the number of places $1 \le i < n$ such that $x_i > x_{i+1}$) and $(ab)(w)$ denotes the number $\mathrm{rise}\, w$ of *rises* of w (i.e., the number of places $1 \le i < n$ such that $x_i < x_{i+1}$).

The classical permutation statistics inv and maj may be written as $(bc-a)+(ca-b)+(cb-a)+(ba)$ and $(a - cb) + (b - ca) + (c - ba) + (ba)$, respectively. This inspired Babson and Steingrímsson to perform a computer search for all statistics that could be thus written, and look for those that appear to be Mahonian. They came up with a list of 18. Some of them turned out to be well-known, and some were new. Yet eight new conjecturally Mahonian statistics were left open. Here we prove four of them.

2. Notations

Recall the usual notations

$$(a; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \text{if } n \ge 1, \end{cases}$$

$$(a; q)_\infty = \lim_{n\to\infty} (a; q)_n = \prod_{n \ge 0} (1 - aq^n),$$

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{i=0}^{n-1} q^i, \qquad [n]_q! = \frac{(q;q)_n}{(1-q)^n} = \prod_{i=1}^{n} [i]_q.$$

A statistic stat on the symmetric group $\mathcal{S}_n$ is said to be *Mahonian*, if for every $n \geq 0$ we have

$$\sum_{w \in \mathcal{S}_n} q^{\mathrm{stat}\, w} = [n]_q!$$

A sequence $\big(A_n(t,q)\big)_{n\geq 0}$ of polynomials in two variables t and q, is said to be *Euler–Mahonian*, if one of the following equivalent conditions holds:

1. For every $n \geq 0$,
$$\frac{1}{(t;q)_{n+1}} A_n(t,q) = \sum_{s\geq 0} t^s ([s+1]_q)^n.$$
2. The exponential generating function for the fractions $\frac{A_n(t,q)}{(t;q)_{n+1}}$ is given by
$$\sum_{n\geq 0} \frac{u^n}{n!} \frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{s\geq 0} t^s \exp(u[s+1]_q).$$
3. The sequence $\big(A_n(t,q)\big)$ satisfies the recurrence relation:

(1) $$(1-q)A_n(t,q) = (1-tq^n)A_{n-1}(t,q) - q(1-t)A_{n-1}(tq,q).$$

4. Let $A_n(t,q) = \sum_{s\geq 0} t^s A_{n,s}(q)$. Then the coefficients $A_{n,s}(q)$ satisfy the recurrence:
$$A_{n,s}(q) = [s+1]_q A_{n-1,s}(q) + q^s [n-s]_q A_{n-1,s-1}(q).$$

Now a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ defined on each symmetric group $\mathcal{S}_n$ $(n \geq 0)$ is said to be *Euler–Mahonian*, if for every $n \geq 0$ we have

$$\sum_{w \in \mathcal{S}_n} t^{\mathrm{stat}_1 w} q^{\mathrm{stat}_2 w} = A_n(t,q).$$

3. Results

Our results are the following:

Theorem 1. *The permutation statistic* $S11 = (a-cb) + 2(b-ca) + (ba)$ *is Mahonian.*

Theorem 2. *The permutation statistic* $S13 = (a-cb) + 2(b-ac) + (ab)$ *is Mahonian.*

Theorem 3. *Let* $S5 = (b-ca) + (c-ba) + (a-bc) + (ab)$. *Then, the pair* $(\mathrm{rise}, S5)$ *is Euler–Mahonian.*

Theorem 4. *Let* $S6 = (ba-c) + (c-ba) + (ac-b) + (ba)$. *Then, the pair* $(\mathrm{des}, S6)$ *is Euler–Mahonian.*

Our Theorems 1, 2, and 4 are the three parts of Conjecture 8 of [1], while Theorem 3 is Conjecture 10 of [1]. It turns out that, thanks to Zeilberger's recent theory of the *Umbral Transfer Matrix Method* [4], the proofs of the first three theorems are completely automatic, using the general Maple package ROTA, together with a new interfacing package PERCY that computes the appropriate Rota operators for what we will call *Markovian Permutation Statistics.*

However, ROTA is useless in the case of $S6$. So proving Theorem 4 still requires the traditional combinatorial method: construct a bijection $w \longmapsto w'$ of $\mathcal{S}_n$ onto itself which has the property that

$$(\mathrm{des}, S6)\, w' = (\mathrm{des}, \mathrm{maj})\, w$$

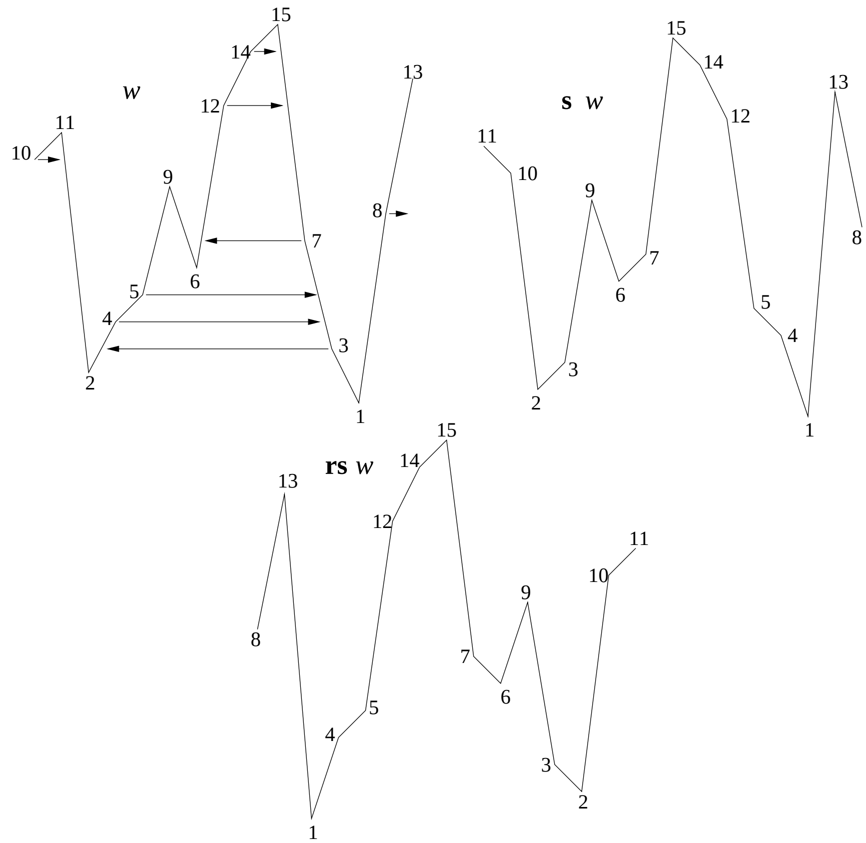

FIGURE 1. $\mathbf{r}\,\mathbf{s}(10, 11, 2, 4, 5, 9, 6, 12, 14, 15, 7, 3, 1, 8, 13) = (8, 13, 1, 4, 5, 12, 14, 15, 7, 6, 9, 3, 2, 10, 11)$.

holds for every $w \in \mathcal{S}_n$.

4. **Proof of Theorem 4**

Instead of the pair (des, maj) we will take another Euler–Mahonian pair (des, mak), where mak is a Mahonian statistic that was introduced by Foata and Zeilberger in [3]. In the Babson–Steingrímsson notation mak reads

$$\mathrm{mak} := (a - cb) + (cb - a) + (ba) + (ca - b).$$

First, the *descent bottom* of a permutation $x_1x_2 \dots x_n$ is defined to be the set $\operatorname{desbot} w$ of all the x_i's such that $2 \le i \le n$ and $x_{i-1} > x_i$. Its cardinality is the number $\operatorname{des} w$ of descents of w.

Next, the word statistics U and V are introduced as follows. Let $y = x_i$ be a letter of the permutation $w = x_1x_2 \dots x_n$. Define

$$U_y(w) = (ca - b)|_{b=y}\, w; \qquad V_y(w) = (b - ac)|_{b=y}\, w.$$

Thus, $U_y(w)$ is the number of adjacent letters x_jx_{j+1} to the left of $y = x_i$ such that $x_j > x_i > x_{j+1}$. The word statitics U and V are then:

$$U(w) = U_1(w)U_2(w) \dots U_n(w); \qquad V(w) = V_1(w)V_2(w) \dots V_n(w).$$

Now, recall the traditional reverse image $\mathbf{r}$, which is an involution that maps each permutation $w = x_1x_2 \dots x_n$ onto $\mathbf{r}\,w = x_nx_{n-1} \dots x_1$. We shall introduce another involution $\mathbf{s}$ of $\mathcal{S}_n$, called the *rise-des-exchange*, which exchanges the rises and the descents of a permutation and keeps peaks and troughs in their original ordering. The involution $\mathbf{s}$ is not explained here, but can be immediately visualized in Fig. 1.

Proposition 1. *The involution* $\mathbf{r\,s}$ *of* $\mathcal{S}_n$ *has the following properties:*

1. $\operatorname{desbot} \mathbf{r\,s}\, w = \operatorname{desbot} w$,
2. $(U, V)\ \mathbf{r\,s}\, w = (V, U)\, w$.

Let Σ-desbot w be the sum of all the letters x_i of the permutation $w = x_1 x_2 \ldots x_n$ which belong to the descent bottom set desbot w.

Proposition 2. *For each permutation* w *we have:*

$$\Sigma\text{-desbot}\, w = \big((a - cb) + (cb - a) + (ba)\big)\, w.$$

Next, we introduce the *complement* to $(n+1)$, denoted by $\mathbf{c}$, that maps each permutation $w = x_1 x_2 \ldots x_n$ onto $\mathbf{c}\, w = (n+1-x_1)(n+1-x_2)\ldots(n+1-x_n)$. Thus the statistic $S6\ \mathbf{r\,c}$ reads

$$S6\ \mathbf{r\,c} = (a - cb) + (cb - a) + (ba) + (b - ca).$$

Taking Proposition 2 into account, we get the expressions:

$$\operatorname{mak} w = \Sigma\text{-desbot}\, w + U_1(w) + \cdots + U_n(w),$$

$$S6\ \mathbf{r\,c} = \Sigma\text{-desbot}\, w + V_1(w) + \cdots + V_n(w).$$

Therefore, Proposition 1 implies the following corollary.

Corollary 1. *The involution* $\mathbf{r\,s}$ *is an involution of* $\mathcal{S}_n$ *having the property:*

$$(\operatorname{des}, \operatorname{mak})\, w = (\operatorname{des}, S6\ \mathbf{r\,c})\ \mathbf{r\,s}\, w.$$

But (des, mak) is Euler–Mahonian, as proved in [3]. Therefore, the pair $(\operatorname{des}, S6\ \mathbf{r\,c})$ is Euler–Mahonian, as well as (des,S6), since we always have $\operatorname{des} \mathbf{r\,c}\, w = \operatorname{des} w$. Hence Theorem 4 is proved.

5. Markovian Permutation Statistics

The *reduction* of a sequence w of n distinct integers, denoted by $\operatorname{red}(w)$, is the permutation obtained by replacing the smallest member by 1, the second-smallest by 2, …, and the largest by n. For example $\operatorname{red}(5\ 8\ 3\ 7\ 4) = 3\ 5\ 1\ 4\ 2$.

A permutation statistic $F : \mathcal{S}_n \to \mathbb{Z}$ is said to be *Markovian*, if there exists a function $h(j, i, n)$ such that

$$F(x_1 \ldots x_n) = F\big(\operatorname{red}(x_1 \ldots x_{n-1})\big) + h(x_{n-1}, x_n, n).$$

A Markovian permutation statistic $F : \mathcal{S}_n \to \mathbb{Z}$ is said to be *nice Markovian* if the above $h(j, i, n)$ can be written as

$$h(j, i, n) = \begin{cases} f(j, i, n) & \text{if } j < i, \\ g(j, i, n) & \text{if } j > i, \end{cases}$$

where f and g are *affine linear functions* of their arguments, i.e., can be written as $ai + bj + cn + d$, for some integers a, b, c, d.

We, and the Maple package PERCY, will only consider nice Markovian statistics. We will denote them by $[f, g, j, i, n]$. For exemple, $\operatorname{inv} = [n - i, n - i, j, i, n]$, $\operatorname{maj} = [0, n-1, j, i, n]$, $\operatorname{des} = [0, 1, j, i, n]$, $\operatorname{rise} = [1, 0, j, i, n]$.

Given a permutation statistic F we are interested in the sequence of polynomials

$$\operatorname{gf}(F)_n(q) = \sum_{w \in \mathcal{S}_n} q^{F(w)} \qquad (n \geq 0).$$

However, in order to take advantage of Markovity, we need to consider the more refined

$$\mathrm{GF}(F)_n(q,z) = \sum_{w=x_1\dots x_n\in\mathcal{S}_n} q^{F(w)} z^{x_n} \qquad (n\geq 0)$$

that also keeps track of the last letter x_n. Now, by using *Rota operators* [4], it is easy to express $\mathrm{GF}(F)_n$ in terms of $\mathrm{GF}(F)_{n-1}$. Let $w' = x'_1 \dots x'_{n-1} = \mathrm{red}(x_1 \dots x_{n-1})$; then

$$\begin{aligned}\mathrm{GF}(F)_n(q,z) &= \sum_{i=1}^{n} z^i \sum_{w\in\mathcal{S}_n;\ x_n=i} q^{F(w)} \\ &= \sum_{j=1}^{n-1} \sum_{w'\in\mathcal{S}_{n-1};\ x'_{n-1}=j} \left(\sum_{i=1}^{j} q^{g(j+1,i,n)} z^i + \sum_{i=j+1}^{n} q^{f(j,i,n)} z^i\right) q^{F(w')}.\end{aligned}$$

Now for $i \leq j \leq n-1$ we introduce the *umbra* $\mathcal{P}$,

$$\mathcal{P}(z^j) = \left(\sum_{i=1}^{j} q^{g(j+1,i,n)} z^i + \sum_{i=j+1}^{n} q^{f(j,i,n)} z^i\right),$$

and we extend by linearity, so that $\mathcal{P}$ is defined on all polynomials of degree less than or equal to $n-1$. In terms of $\mathcal{P}$, we have the very simple recurrence:

$$\mathrm{GF}(F)_n(q,z) = \mathcal{P}\big(\mathrm{GF}(F)_{n-1}(q,z)\big).$$

Maple can compute the umbra automatically. All the users have to enter is f and g, and PERCY would convert it to the Markovian notation.

6. Proof of Theorem 1

Using PERCY and ROTA we get that the umbra $\mathcal{P}$ linking $\mathrm{GF(S11)}_{n-1}(q,z)$ to $\mathrm{GF(S11)}_n(q,z)$ maps the polynomial $a(z)$ onto

$$\frac{z^{n+1}a(1) - za(z)}{z-1} + \frac{z\left(a(qz) - a(q^2)\right)}{z-q}.$$

Hence $b_n(z) = \mathrm{GF(S11)}_n(q,z)$ satisfies the functional recurrence

$$b_n(z) = \frac{z^{n+1}b_{n-1}(1) - zb_{n-1}(z)}{z-1} + \frac{z\left(b_{n-1}(qz) - b_{n-1}(q^2)\right)}{z-q},$$

with the initial condition $b_1(z) = z$. But if we guess (and if we check) that the sequence

$$c_n(z) = z\frac{z^n - q^n}{z-q}[n-1]_q!$$

satisfies the same recurrence, we obtain that $b_n(z) = c_n(z)$, and finally that $b_n(1) = c_n(1) = [n]_q!$

7. Proof of Theorem 2

Using PERCY and ROTA we get that the umbra $\mathcal{P}$ linking $\mathrm{GF(S13)}_{n-1}(q,z)$ to $\mathrm{GF(S13)}_n(q,z)$ maps the polynomial $a(z)$ onto

$$\frac{z\big(a(zq) - a(1)\big)}{qz-1} + \frac{zqa(z) - q^{2n+1}z^{n+1}a(q^{-2})}{1-zq^2}.$$

Hence $d_n(z) = \mathrm{GF(S13)}_n(q,z)$ satisfies the functional recurrence

$$d_n(z) = \frac{z\big(d_{n-1}(zq) - d_{n-1}(1)\big)}{qz-1} + \frac{zqd_{n-1}(z) - q^{2n+1}z^{n+1}d_{n-1}(q^{-2})}{1-zq^2},$$

with the initial condition $d_1(z) = z$. But if we guess (and if we check) that the sequence

$$e_n(z) = z\frac{(1-z^nq^n)}{1-qz}[n-1]_q!$$

satisfies the same recurrence, we obtain that $d_n(z) = e_n(z)$, and finally that $d_n(1) = e_n(1) = [n]_q!$.

8. Proof of Theorem 3

PERCY can compute the Umbra multi-statistics, when the generating function is the weight-enumerator of $\mathcal{S}_n$ according to the weight

$$\mathrm{weight}(w) = z^{x_n}\prod_{j=1}^{r} q_j^{F_j(w)},$$

where $w = x_1 \ldots x_n$ and $F_1(w), \ldots, F_r(w)$ are several nice Markovian permutation statistics. Define

$$A_n(t,q;z) = \sum_{w\in\mathcal{S}_n} t^{\mathrm{des}\, w}q^{\mathrm{maj}\, w}z^{x_n}, \qquad B_n(t,q;z) = \sum_{w\in\mathcal{S}_n} t^{\mathrm{rise}\, w}q^{\mathrm{S5}\, w}z^{x_n}.$$

PERCY and ROTA compute the following functional recurrences

$$A_n(t,q;z) = \frac{z(1-tq^{n-1})A_{n-1}(t,q;z) - z(z^n - tq^{n-1})A_{n-1}(t,q;1)}{1-z}; \tag{2}$$

$$B_n(t,q;z) = \frac{z(1-tq^n)B_{n-1}(t,q;z) - z(1-tz^n)B_{n-1}(t,q;q)}{z-q}.$$

By comparing the two functional recurrences, we guess and we verify that

$$B_n(t,q;z) = q^{-n}z^{n+1}A_n(tq,q;q/z).$$

Hence $B_n(t,q;1) = q^{-n}A_n(tq,q;q)$. By plugging $t = tq,\ \ z = q$ into Eq. (2), we get that

$$A_n(tq,q;q) = q^n\frac{(1-tq^n)A_{n-1}(t,q;1) - q(1-t)A_{n-1}(tq,q;1)}{1-q}.$$

But, this equals $q^nA_n(t,q)$ by Eq. (1). And we have proved that $B_n(t,q;1) = A_n(t,q;1) = A_n(t,q)$.

The input and output files of PERCY can be downloaded from

`http://www.math.temple.edu/~zeilberg/programs.html`.

Bibliography

[1] Babson (Eric) and Steingrímsson (Einar). – Generalized permutation patterns and a classification of the Mahonian statistics. *Séminaire Lotharingien de Combinatoire*, vol. 44, n° B44b, 2000. – 18 pages. Available from `http://www.mat.univie.ac.at/~slc/`.

[2] Foata (D.) and Zeilberger (D.). – Babson-Steingrímsson statistics are indeed Mahonian (and sometimes even Euler-Mahonian). – To appear in *Advances in Applied Mathematics*.

[3] Foata (Dominique) and Zeilberger (Doron). – Denert's permutation statistic is indeed Euler-Mahonian. *Studies in Applied Mathematics*, vol. 83, n° 1, 1990, pp. 31–59.

[4] Zeilberger (Doron). – The umbral transfer-matrix method. I. Foundations. *Journal of Combinatorial Theory. Series A*, vol. 91, n° 1-2, 2000, pp. 451–463. – In memory of Gian-Carlo Rota.