<u>Variations on the Sequence of Apéry Numbers</u>

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In the early 1990's, Doron Zeilberger and Herbert Wilf, developped a new methodology for symbolic summation and integration (Wilf, Herbert S. and Zeilberger, Doron (1992):An algorithmic proof theory for hypergeometric (ordinary and ''q'') multisum/integral identities, *Inventiones Mathematicae*, **108**:575-633). One of the most famous successes of this so-called ''WZ-method'' has been to provide a computer proof of the combinatorial identity

$$\sum_{k=0}^{n} \operatorname{binomial}(n, k)^2 \operatorname{binomial}(n+k, k)^2 =$$

$$\sum_{k=0}^{n} \operatorname{binomial}(n, k) \operatorname{binomial}(n+k, k) \left(\sum_{j=0}^{k} \operatorname{binomial}(k, j)^3\right),$$

and to prove that the sequence of these numbers a_n satisfies the second order recurrence equation

$$(n+2)^3 u_{n+2} - ((n+2)^3 + (n+1)^3 + 4(2n+3)^3) u_{n+1} + (n+1)^3 u_n = 0.$$

Proving this recurrence was a crucial step of Apéry's proof for the irrationality of

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

On the other hand, the identity itself stems from a number-theoretic question raised by Schmidt in (Schmidt, Asmus L. (1990): Generalized Legendre polynomials, *J. reine angew. Math.*, **404** :192-202).

Several proofs of the identity above, that relates the Apéry numbers a_n to the Franel numbers

$$f_n = \sum_{k=0}^{n} \operatorname{binomial}(n, k)^3,$$

were given in (Strehl, Volker (1994): Binomial Identities, Combinatorial and Algorithmic Aspects, *Discrete Math.*, **136**:309-346). One of them in particular is based on Zeilberger's algorithm for hypergeometric summation, and yields the recurrence equation above as a by-product. In the following sections, we first recall how Apéry was led to the identity, borrowing from Van der Poorten's report (Van der Poorten, Alfred (1979): A Proof that Euler missed... Apéry's Proof of the Irrationality of $\zeta(3)$, *Math. Intelligencer*, **1**:195-203); we next give a proof for both results using our package <u>Mgfun</u>, and finally exploit the recurrence equation to beat *Maple* computing many digits of $\zeta(3)$.

Sketch of Apéry's Proof

Apéry's first remark is that the double sequence

$$c_{n,k} = \left(\sum_{m=1}^{n} \frac{1}{m^3}\right) + \left(\sum_{m=1}^{k} \frac{(-1)^{(m+1)}}{2\, m^3 \operatorname{binomial}(n, m) \operatorname{binomial}(n+m, m)}\right)$$

tends to $\zeta(3)$ uniformly in k for k in $(1 .. \infty)$ when n tends to infinity. This stems from the alternating series being uniformly bounded by $\frac{1}{n}$. However, the convergence of this series is not strong enough so as to show the irrationality of $\zeta(3)$. Apéry used summation methods to accelerate the convergence. Namely, define

$$a_n = \sum_{k=0}^{n} \operatorname{binomial}(n, k)^2 \operatorname{binomial}(n+k, k)^2 c_{n,k} \quad \text{and}$$

$$b_n = \sum_{k=0}^{n} \operatorname{binomial}(n, k)^2 \operatorname{binomial}(n+k, k)^2,$$

then $\frac{a_n}{b_n}$ also tends to $\zeta(3)$. Here appears the crucial recurrence of Apéry: one remarks that it is satisfied by both sequences a and b, with initial conditions

$$a_0 = 0, \quad a_1 = 6, \quad \text{and} \quad b_0 = 1, \quad b_1 = 5.$$

By a number-theoretic argument, it follows from this recurrence that

$$\zeta(3) - \frac{a_n}{b_n} = \mathrm{O}(q_n^{(-1+\delta)}),$$

with

$$q_n = 2 \operatorname{lcm}(1, 2, .. , n)^3 b_n \quad \text{and} \quad \delta = \frac{4 \ln(1+\sqrt{2}) - 3}{4 \ln(1+\sqrt{2}) + 3}, \text{ which is positive.}$$

This is sufficient to prove that $\zeta(3)$ is irrational, and yields an irrationality measure of at least

$$1 + \frac{1}{\delta} = 13.417820 ..$$

Proof of Apéry's Recurrence

In this session, we chiefly use the user-oriented package Mgfun.

```
> with(Mgfun);
```

$$[\mathit{diag\_of\_sys}, \mathit{int\_of\_sys}, \mathit{pol\_to\_sys}, \mathit{sum\_of\_sys}, \mathit{sys*sys}, \mathit{sys+sys}]$$

Recurrence for the Left-Hand Side

We first prove that the Apéry numbers, as defined by the left-hand side

$$a_n = \sum_{k=0}^{n} \operatorname{binomial}(n, k)^2 \operatorname{binomial}(n+k, k)^2,$$

satisfy the announced recurrence.

The summand

$> f := \operatorname{binomial}(n, k)^2 \operatorname{binomial}(n+k, k)^2$

satisfies both following equations:

```
> h(n+1,k)/h(n,k)=factor(normal(subs(n=n+1,f)/f,expanded));
```

$$\frac{h(n+1,k)}{h(n,k)} = \frac{(n+1+k)^2}{(-n-1+k)^2}$$

```
> h(n,k+1)/h(n,k)=factor(normal(subs(k=k+1,f)/f,expanded));
```

$$\frac{h(n,k+1)}{h(n,k)} = \frac{(n+1+k)^2\,(-n+k)^2}{(k+1)^4}$$

This yields the following system

```
> sys:=collect(map(numer,map(eq->op(1,eq)-op(2,eq),{"",""})),
  h);
```

$$sys := \{ (n^2+2\,n-2\,n\,k+1-2\,k+k^2)\,h(n+1,k)+(-2\,n\,k-1-n^2-2\,n-2\,k-k^2)\,h(n,k)$$
$$,(-n^4-2\,n^3-n^2+2\,n^2\,k+2\,n^2\,k^2+2\,n\,k+2\,n\,k^2-k^2-2\,k^3-k^4)\,h(n,k)$$
$$+(k^4+4\,k^3+6\,k^2+4\,k+1)\,h(n,k+1)\}$$

where each element *expr* in the set denotes the equation $expr = 0$. The definite summation over k in $(0, n)$ is performed by the following call to Mgfun[sum_of_sys]:

```
> sum_of_sys(sys,k=-infinity..infinity,takayama_algo);
```

$$\{(n^3+3\,n^2+3\,n+1)\,h(n)+(-34\,n^3-231\,n-153\,n^2-117)\,h(n+1)$$
$$+(n^3+6\,n^2+12\,n+8)\,h(n+2)\}$$

```
> rec[left]:=op(collect(",h,factor));
```

$$rec_{left} := (n+1)^3\,h(n)+(n+2)^3\,h(n+2)-(2\,n+3)\,(17\,n^2+51\,n+39)\,h(n+1)$$

This is the announced recurrence in disguise.

Recurrence for the Inner Sum

We next compute the right-hand side, beginning with the Franel numbers f_n given by

$$f_n = \sum_{k=0}^{n} \mathrm{binomial}(n,k)^3.$$

We first obtain a system to describe the summand:

> $f := \mathrm{binomial}(n,k)^3$

> $\frac{h(n+1,k)}{h(n,k)} - \mathrm{factor}\left(\mathrm{normal}\left(\frac{\mathrm{subs}(n=n+1,f)}{f}, expanded\right)\right)$

> $\frac{h(n,k+1)}{h(n,k)} - \mathrm{factor}\left(\mathrm{normal}\left(\frac{\mathrm{subs}(k=k+1,f)}{f}, expanded\right)\right)$

```
> sys:=collect(map(numer,{"",""}),h);
```

$$sys := \{(-n^3+3\,n^2\,k-3\,n\,k^2+k^3)\,h(n,k)+(k^3+3\,k^2+3\,k+1)\,h(n,k+1),$$
$$(-n^3-3\,n^2+3\,n^2\,k-3\,n+6\,n\,k-3\,n\,k^2-1+3\,k-3\,k^2+k^3)\,h(n+1,k)$$
$$+(n^3+3\,n^2+3\,n+1)\,h(n,k)\}$$

Summing, we get a recurrence for the Franel numbers f_n.

```
> sys:=sum_of_sys(sys,k=-infinity..infinity,natural_boundari
  es);
```

$$sys := \{(-56-136\,n-104\,n^2-24\,n^3)\,h(n)+(3\,n^3+22\,n^2+51\,n+36)\,h(n+3)$$

$+ (-45\, n^3 - 240 - 240\, n^2 - 419\, n)\, \mathrm{h}(n+1)$
$+ (-18\, n^3 - 148 - 114\, n^2 - 232\, n)\, \mathrm{h}(n+2)\, \}$

System for the Right-Hand Product

Let us multiply the f_k by binomial(n, k) binomial$(n+k, k)$. To do so, we prepare two systems of recurrence equations, one for each factor.

The following system describes f_k, which is independent from n.

```
> sys1:={h(n+1,k)-h(n,k)} union eval(subs(h=proc(k) h(n,k)
  end,subs(n=k,sys)));
```

$$sys1 := \{\, \mathrm{h}(n+1, k) - \mathrm{h}(n, k),\, (-56 - 136\, k - 104\, k^2 - 24\, k^3)\, \mathrm{h}(n, k) + (3\, k^3 + 22\, k^2 + 51\, k + 36)\, \mathrm{h}(n, k+3) + (-45\, k^3 - 240 - 240\, k^2 - 419\, k)\, \mathrm{h}(n, k+1) + (-18\, k^3 - 148 - 114\, k^2 - 232\, k)\, \mathrm{h}(n, k+2)\, \}$$

We next obtain a system that describes binomial(n, k) binomial$(n+k, k)$, the weight to multiply with:

$> f := \mathrm{binomial}(n, k)\, \mathrm{binomial}(n+k, k)$

$> \dfrac{\mathrm{h}(n+1, k)}{\mathrm{h}(n, k)} - \mathrm{factor}\left(\mathrm{normal}\left(\dfrac{\mathrm{subs}(n = n+1, f)}{f}, expanded\right)\right)$

$> \dfrac{\mathrm{h}(n, k+1)}{\mathrm{h}(n, k)} - \mathrm{factor}\left(\mathrm{normal}\left(\dfrac{\mathrm{subs}(k = k+1, f)}{f}, expanded\right)\right)$

```
> sys2:=collect(map(numer,{"",""}),h);
```

$$sys2 := \{\, (-n^2 - n + k + k^2)\, \mathrm{h}(n, k) + (k^2 + 2\, k + 1)\, \mathrm{h}(n, k+1),\, (-n - 1 + k)\, \mathrm{h}(n+1, k) + (n + 1 + k)\, \mathrm{h}(n, k)\, \}$$

We finally perform the product by a call to Mgfun[‘sys*sys‘]:

```
> sys:=`sys*sys`(sys1,sys2);
```

$$sys := \{\, (-n - 1 - k)\, \mathrm{h}(n, k) + (n + 1 - k)\, \mathrm{h}(n+1, k),\, (-672\, k + 672\, n - 272\, k^6 + 224\, n^2 + 2304\, n\, k - 2528\, k^2 + 1608\, n^2\, k + 2664\, n^2\, k^2 + 3048\, n\, k^2 - 3816\, k^3 - 2960\, k^4 + 600\, k^4\, n^2 + 72\, k^5\, n - 280\, n^4 - 840\, n^3 + 1944\, k^3\, n + 72\, n^5\, k + 56\, n^6 - 384\, n^4\, k^2 - 1368\, n^3\, k + 168\, n^5 + 24\, n^6\, k - 768\, n^3\, k^2 - 624\, n^4\, k - 72\, k^3\, n^4 - 144\, k^3\, n^3 + 600\, k^4\, n - 24\, k^7 + 1872\, k^3\, n^2 - 1248\, k^5 + 72\, k^5\, n^2)\, \mathrm{h}(n, k) + (2880 + 11748\, k - 1920\, n + 600\, k^6 - 1680\, n^2 - 5272\, n\, k + 20132\, k^2 - 4853\, n^2\, k - 5512\, n^2\, k^2 - 5752\, n\, k^2 + 18817\, k^3 + 10372\, k^4 - 840\, k^4\, n^2 - 90\, k^5\, n + 240\, n^4 + 480\, n^3 - 3118\, k^3\, n + 240\, n^4\, k^2 + 838\, n^3\, k + 480\, n^3\, k^2 + 419\, n^4\, k + 45\, k^3\, n^4 + 90\, k^3\, n^3 - 840\, k^4\, n + 45\, k^7 - 3073\, k^3\, n^2 + 3374\, k^5 - 90\, k^5\, n^2)\, \mathrm{h}(n, k+1) + (-17384\, k^2 - 1798\, k^5 + 1532\, n^2\, k^2 + 18\, k^5\, n^2 + 18\, k^5\, n + 760\, k^3\, n^2 + 1532\, n\, k^2 + 1520\, n^2\, k + 186\, k^4\, n + 760\, k^3\, n + 592\, n + 1520\, n\, k - 18\, k^7 - 276\, k^6 - 13740\, k^3 + 592\, n^2 - 12080\, k - 3552 - 6448\, k^4 + 186\, k^4\, n^2)\, \mathrm{h}(n, k+2) + (-5184\, k^2 - 52\, k^6 - 3\, k^7 - 3675\, k^3 - 1540\, k^4 - 3996\, k - 1296 - 382\, k^5)\, \mathrm{h}(n, k+3) \}$$

Recurrence for the Right-Hand Side

We obtain a recurrence for the right-hand side by applying Mgfun[sum_of_sys] on the system computed in the previous section.

```
> sum_of_sys(sys,k=-infinity..infinity,takayama_algo);
```

$$\{(-112-466\,n-775\,n^2-295\,n^4-655\,n^3-6\,n^6-67\,n^5)\,\mathrm{h}(n)$$
$$+(37042\,n^3+14280+204\,n^6+13611\,n^4+2612\,n^5+55685\,n^2+43966\,n)\,\mathrm{h}(n+1)$$
$$+(6\,n^6+113\,n^5+870\,n^4+3495\,n^3+7700\,n^2+8784\,n+4032)\,\mathrm{h}(n+4)+$$
$$(-88200-92178\,n^3-189205\,n^2-204\,n^6-3508\,n^5-202734\,n-24811\,n^4)\,\mathrm{h}(n+3)$$
$$+(-57976\,n^3-77840-140445\,n^2-167086\,n-11775\,n^4-942\,n^5)\,\mathrm{h}(n+2)\}$$

This recurrence is different from the one obtained for the left-hand side.

```
> rec[right]:=op(collect(",h,factor));
```

$$rec_{right} := -(2\,n+7)\,(2\,n+3)\,(n+3)\,(51\,n^3+469\,n^2+1418\,n+1400)\,\mathrm{h}(n+3)$$
$$+(2\,n+3)\,(3\,n+7)\,(n+3)\,(n+4)^3\,\mathrm{h}(n+4)$$
$$-(3\,n+8)\,(2\,n+7)\,(n+2)\,(n+1)^3\,\mathrm{h}(n)$$
$$-(2\,n+5)\,(471\,n^4+4710\,n^3+17213\,n^2+27190\,n+15568)\,\mathrm{h}(n+2)$$
$$+(2\,n+7)\,(2\,n+3)\,(n+2)\,(51\,n^3+296\,n^2+553\,n+340)\,\mathrm{h}(n+1)$$

We thus need more work to prove that both sides agree.

Final Proof of the Identity and of the Second Order Recurrence

Let h_n be any solution of the second order recurrence which has been obtained for the left-hand side:

```
> rec[left];
```

$$(n+1)^3\,\mathrm{h}(n)+(n+2)^3\,\mathrm{h}(n+2)-(2\,n+3)\,(17\,n^2+51\,n+39)\,\mathrm{h}(n+1)$$

Thus:

```
> h(n+2)=collect(solve(",h(n+2)),h,normal);
```

$$\mathrm{h}(n+2) = -\frac{(n^3+3\,n^2+3\,n+1)\,\mathrm{h}(n)}{n^3+6\,n^2+12\,n+8}+\frac{(34\,n^3+153\,n^2+231\,n+117)\,\mathrm{h}(n+1)}{n^3+6\,n^2+12\,n+8}$$

```
> subs(n=n+1,");
```

$$\mathrm{h}(n+3) = -\frac{((n+1)^3+3\,(n+1)^2+3\,n+4)\,\mathrm{h}(n+1)}{(n+1)^3+6\,(n+1)^2+12\,n+20}$$
$$+\frac{(34\,(n+1)^3+153\,(n+1)^2+231\,n+348)\,\mathrm{h}(n+2)}{(n+1)^3+6\,(n+1)^2+12\,n+20}$$

and

```
> subs(n=n+1,");
```

$$\mathrm{h}(n+4) = -\frac{((n+2)^3+3\,(n+2)^2+3\,n+7)\,\mathrm{h}(n+2)}{(n+2)^3+6\,(n+2)^2+12\,n+32}$$
$$+\frac{(34\,(n+2)^3+153\,(n+2)^2+231\,n+579)\,\mathrm{h}(n+3)}{(n+2)^3+6\,(n+2)^2+12\,n+32}$$

Then, h_n also solves the recurrence for the right-hand side:

```
> collect(subs(",""",""",rec[right]),h,normal);
```

$$0$$

At this point, we have proved that both sides of the equation satisfy the same recurrence of order 4. To prove the announced equality, we simply need to check 4 initial conditions, since the leading coefficient of the recurrence of order 4,

```
> coeff(rec[right],h(n+3));
```

$$-(2\,n+7)\,(2\,n+3)\,(n+3)\,(51\,n^3+469\,n^2+1418\,n+1400)$$

never vanishes for non-negative n. Now the proof of the identity

$$> \; eq := \sum_{k=0}^{n} \mathrm{binomial}(n, k)^2 \, \mathrm{binomial}(n+k, k)^2 = \sum_{k=0}^{n} \mathrm{binomial}(n, k) \, \mathrm{binomial}(n+k, k) \left(\sum_{j=0}^{k} \mathrm{binomial}(k, j)^3 \right)$$

simplify follows from

```
> eval(subs(n=0,Sum=add,eq));
```

$$1 = 1$$

```
> eval(subs(n=1,Sum=add,eq));
```

$$5 = 5$$

```
> eval(subs(n=2,Sum=add,eq));
```

$$73 = 73$$

```
> eval(subs(n=3,Sum=add,eq));
```

$$1445 = 1445$$

Therefore, the Apéry numbers also satisfy the announced second order recurrence.

Computation of $\zeta(3)$ Using Standard Maple and the Holonomic Approach

Standard Maple

Maple has numerical routines for almost all special functions it knows about. Here is the corresponding calculation for $\zeta(3)$.

```
> ti[0]:=time():
> Z3[standard]:=evalf(Zeta(3),332);
```

$Z3_{standard}$:= 1.2020569031595942853997381615114499907649862923404988817922\
7155534183820578631309018645587360933525814619915779526071941849199\
5998673283213776396837207900161453941782949360066719191575522242494\
2439615639096641032911590957809655146512799184051057152559880154371\
0978110203982753256678760352233698494166181105701471577863949973752378\
53

(Note: printed line breaks as shown)

```
> ti[standard]:=time()-ti[0];
```

$$ti_{standard} := 38.315$$

Holonomic Approach

We compute an approximation of $\zeta(3)$ using Apéry's recurrence. More precisely, we compute it as $\frac{a_{200}}{b_{200}}$. (Remember that $\frac{a_n}{b_n}$ tends to $\zeta(3)$.)

```
> ti[0]:=time():
> N:=200:
```

To do so, we use the gfun package by Salvy and Zimmermann (Salvy, Bruno and Zimmermann, Paul (1994): Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Trans. Math. Software*, **20**(2):163-177).

```
> with(gfun);
```

[*Laplace*, *algebraicsubs*, *algeqtodiffeq*, *algeqtoseries*, *algfuntoalgeq*, *borel*, *cauchyproduct*, *diffeq*diffeq*, *diffeq+diffeq*, *diffeqtohomdiffeq*, *diffeqtorec*, *guesseqn*, *guessgf*, *hadamardproduct*, *holexprtodiffeq*, *invborel*, *listtoalgeq*, *listtodiffeq*, *listtohypergeom*, *listtolist*, *listtoratpoly*, *listtorec*, *listtoseries*, *listtoseries/Laplace*, *listtoseries/egf*, *listtoseries/lgdegf*, *listtoseries/lgdogf*, *listtoseries/ogf*, *listtoseries/revegf*, *listtoseries/revogf*, *maxdegcoeff*, *maxdegeqn*, *maxordereqn*, *mindegcoeff*, *mindegeqn*, *minordereqn*, *optionsgf*, *poltodiffeq*, *poltorec*, *ratpolytocoeff*, *rec*rec*, *rec+rec*, *rectodiffeq*, *rectohomrec*, *rectoproc*, *seriestoalgeq*, *seriestodiffeq*, *seriestohypergeom*, *seriestolist*, *seriestoratpoly*, *seriestorec*, *seriestoseries*]

The *gfun* package provides us with a routine for transforming a recurrence equation like

> $eq := n^3\,\mathrm{u}(n) - (34\,n^3 - 51\,n^2 + 27\,n - 5)\,\mathrm{u}(n-1) + (n-1)^3\,\mathrm{u}(n-2)$

into a procedure. Each of the following procedures encodes the calculation of a sequence u_n given by the equation *eq* and its initial values u_0 and u_1.

```
> A:=rectoproc({eq,u(0)=0,u(1)=6},u(n));
```

A := **proc**(*n*)
local *i*, *u0*, *u1*, *u2*;
    u0 := 0;
    u1 := 6;
    for *i* **from** 2 **to** $n-1$ **do**
        $u2 := -(-u0 + 5*u1 + (3*u0 - 27*u1 + (-3*u0 + 51*u1 + (u0 - 34*u1)*i)*i)*i) / i\verb|^|3;$
        u0 := *u1*;
        u1 := *u2*
    od;
    $-(-u0 + 5*u1 + (3*u0 - 27*u1 + (-3*u0 + 51*u1 + (u0 - 34*u1)*n)*n)*n) / n\verb|^|3$
end

$\#(0) = 0$

$\#(1) = 6$

```
> B:=rectoproc({eq,u(0)=1,u(1)=5},u(n));
```

$B :=$ **proc**(n)

local i, $u0$, $u1$, $u2$;

$u0 := 1$;

$u1 := 5$;

for i **from** 2 **to** $n-1$ **do**

$u2 := -(-u0 + 5*u1$
$+ (3*u0 - 27*u1 + (-3*u0 + 51*u1 + (u0 - 34*u1)*i)*i)*i) / i\verb|^|3$;

$u0 := u1$;

$u1 := u2$

od;

$-(-u0 + 5*u1 + (3*u0 - 27*u1 + (-3*u0 + 51*u1 + (u0 - 34*u1)*n)*n)*n) /$
$n\verb|^|3$

end

$\#(0) = 1$

$\#(1) = 5$

Compute a_{200} and b_{200}:

```
> a:=A(N);
```

$a :=$ 27120214525404796485818105490387356973722905867369270689468377839\
4422456140389412839608794880972692282601501783604410962338497164259\
5840154558316945119976103968513466948860251984092965649522022675341 8\
9608977318095862025559812472136621665907699556970385597872501757096 0\
7313978057228741507859352529064023054719543240473562743519551865249 3\
9964519767521192385545620750079011320884150908528361628960032733964 5\
2698834961720547642429009932410383314354559768468491976134587879557 0\
6105624983505466901079580207803856159353784285507758452163223732388 2\
7320967799557235555299 / 174889846834181722648388614514189413474473\
3389289598894911920918883191853710795996168314717208968684112179195 9\
4916014994356846789477504850137839700327180750449621467859666093774 3\
4506323800407484223695545905474439344164800219410494146401865604654 7\
25939200000000

```
> b:=B(N);
```

$b :=$ 129004094966392642601682058459386547279545028730430018466222011 73\
4254061922238148079190892478247908964984869330303027456102143438112 3\
7450992762875529150161873461645140207045923747436598217143513780548 0\
4739575921340322343759008895642338051968350490635346050623308076965 9\

4554733147636155936662958465207425

A priori, it is not clear how many digits we can guarantee.

```
> P:=round(evalf(ln(a)/ln(10))*1.1);
```

$$P := 332$$

```
> Z3[holonomy]:=evalf(a/b,P);
```

$Z3_{holonomy}$:= 1.2020569031595942853997381615114499907649862923404988817922\
2715553418382057863130901864558736093352581461991577952607194184919\
5998673283213776396837207900161453941782949360066719191575522242494\
2439615639096641032911590957809655146512799184051057152559880154371\
0978110203982753256678760352233698494166181105701471577863949973752378\
53

The time used is

```
> ti[holonomy]:=time()-ti[0];
```

$$ti_{holonomy} := 7.286$$

Comparison

The holonomic approach is several time faster.

```
> ti[standard]/ti[holonomy];
```

$$5.258715344$$

Moreover, this ratio would increase with the accuracy of the calculations. In this session, we have obtained $\zeta(3)$ up to 332 digits.

```
> Z3[standard]-Z3[holonomy];
```

$$0$$

However, *Maple* would be able to compute $\zeta(z)$ for *any* complex value z, while the holonomic approach only works for $z = 3$.