Non Overlapping Partitions, Continued Fractions, Bessel Functions, and a Divergent Series

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Abstract. The counting sequence of a special class of set partitions leads to special numbers called Bessel numbers. The corresponding ordinary generating function has a simple continued fraction expansion related to Bessel functions. We determine here the asymptotic form of Bessel numbers and discuss their relation to Bell numbers. The estimation problem is of some methodological interest as it requires to find the asymptotic form of coefficients in an asymptotic but divergent expansion.

1 Introduction

The number of ways of partitioning an n–set into equivalence classes is the familiar Bell number B_n [2], with exponential generating function

$$\hat{B}(z) \equiv \sum_{n\geq 0} B_n \frac{z^n}{n!} = e^{e^z-1}. \tag{1}$$

For reference, the first few Bell numbers are 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147. Expanding the generating function yields an exact infinite sum for Bell numbers,

$$B_n = e^{-1} \sum_{k\geq 0} \frac{k^n}{k!}, \tag{2}$$

a formula first derived by Dobinski in 1877. Finally, using a combinatorial theory of algebraic continued fraction, Flajolet [7] derived for the ordinary

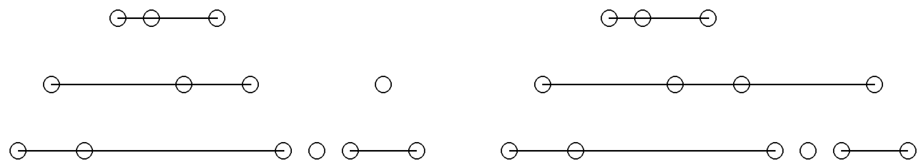

Figure 1. Two types of set partitions.

—and divergent— generating function, a formal expansion

$$B(z) \equiv \sum_{n\geq 0} B_n z^n = \cfrac{1}{1 - 1\cdot z - \cfrac{1\cdot z^2}{1 - 2\cdot z - \cfrac{2\cdot z^2}{1 - 3\cdot z - \cfrac{3\cdot z^2}{\cdots}}}}, \qquad (3)$$

a formula closely related to classical results on Poisson–Charlier polynomials.

This paper is concerned with a special class of partitions, called *non–overlapping* partitions (NOP's). It is customary and convenient to identify the underlying n–set to the integer interval $[1..n]$. With the implied order structure, two blocks (classes) γ, δ *overlap* if

$$\min(\gamma) < \min(\delta) < \max(\gamma) < \max(\delta).$$

For instance, in partition $\varpi = \{\{1,3,4\},\{2,5\}\}$, the two blocks $\gamma = \{1,3,4\}$ and $\delta = \{2,5\}$ overlap. A partition is then called *non–overlapping* if no pair of classes γ, δ overlaps. Thus

$$\varpi = \{\{1,3,9\},\{2,6,8\},\{4,5,7\},\{10\},\{11,13\},\{12\}\} \qquad (4)$$

is non–overlapping. In other terms, call *support* of block γ, the interval $[\min(\gamma), \max(\gamma)]$. In a NOP, supports have a nested structure: For any two (block) supports, either they are disjoint or one covers (contains) the other[1].

[1] A somewhat related notion also appears in the literature [13]: Two classes γ, δ in a partition "cross" if $\exists\, x, y \in \gamma$ and $\exists\, z, t \in \delta$ such that $x < z < y < t$. Kreweras established that the number of non–crossing partitions of size n is the familiar Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus non–overlapping partitions lie somewhere between non–crossing and unconstrained partitions. We shall actually prove that $C_n \ll B_n^* \ll B_n$, with B_n^* the number of NOP's of size n.

Let B_n^* denote the number of NOP's over n elements. These numbers entertain close relations with Bessel functions. For this reason, we chose to call the B_n^*, *Bessel numbers*. (In the same spirit, Bell numbers are sometimes called exponential numbers because of the shape of their generating function.) Our purpose here is to obtain an asymptotic form for Bessel numbers, our main results being summarized by Theorem 1 and Proposition 4.

Direct enumeration shows that the sequence of Bessel numbers starts with

1, 1, 2, 5, 14, 43, 143, 509, 1922, 7651, 31965, 139685, 636712, 3020203, 14878176, 75982829, 401654560, 2194564531, 12377765239, 71980880885.

(The difference between $B_4 = 15$ and $B_4^* = 14$ is due to the unique overlapping partition for $n = 4$, namely $\varpi = \{\{1,3\},\{2,4\}\}$.)

At present, the authors do not know of a simple exponential generating function that would be the analogue of Eq. (1). In more combinatorial terms, NOP's do not decompose as easily as unconstrained partitions that are simply "sets of sets of atoms". (In fact the shape of our results strongly suggests that no simple expression is available.)

The starting point for our treatment is a continued fraction analogue of (3), namely

$$B^*(z) \equiv \sum_{n\geq 0} B_n^* z^n = \cfrac{1}{1 - 1\cdot z - \cfrac{1\cdot z^2}{1 - 2\cdot z - \cfrac{1\cdot z^2}{1 - 3\cdot z - \cfrac{1\cdot z^2}{\cdots}}}}, \tag{5}$$

which derives painlessly from earlier combinatorial works. The difference between (5) and (3) is that numerators are reduced from the integer sequence $1, 2, 3, \ldots$ to $1, 1, 1, \ldots$, a reflection of the fact that NOP's have asymptotic density 0 amongst the class of all partitions of size n.

A difficulty however awaits us, since $B^*(z)$ is defined by (5) only as a formal power series expansion. Series $B^*(z)$ is a purely *divergent* series: it has radius of convergence 0.

Our first step will therefore be to attach continued fraction (5) to special functions of analysis. It turns out that it is expressible in terms of the Bessel function J,

$$J_\nu(x) = \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}(\frac{x}{2})^{2m+\nu}, \tag{6}$$

and is also closely related to the Lommel polynomials [1]. We find the identity

$$\frac{J_{\nu-1}(2)}{\nu J_\nu(2)} \sim 1 - \sum_{n\geq 0} B_n^* \frac{(-1)^n}{\nu^{n+2}}, \tag{7}$$

in the sense that the right hand side represents asymptotically the function on the left, as $\nu \to +\infty$.

The problem is now to find the *asymptotic* form of coefficients in an *asymptotic* (and divergent) series. No Cauchy theorem will do for that purpose.

Using an intuition that goes back at least to Mellin, we shall try to relate the expansion of the function in (6) as $\nu \to +\infty$ to the geometry of its poles as $\nu \to -\infty$. The key to doing this is a Mittag–Leffler expansion which generalizes, for meromorphic functions, the familiar partial fraction expansion of rational functions.

In this manner, we obtain an exact form for the B_n^* which is however rather useless as such: It is expressed in terms of the indices ν of Bessel functions J_ν that admit $x = 2$ as a root. (Studies on zeroes of Bessel functions usually assume that parameter ν is kept fixed, and let x vary.) But, Nature helping, the geometry of these numbers is itself asymptotically simple, and we are lead to an asymptotic equivalent of B_n^*,

$$B_n^* \sim \sum_{k\geq 0} \frac{k^{n+2}}{(k!)^2}, \tag{8}$$

an asymptotic analogue to Dobinski's formula (2). This is our main result (Theorem 1).

One of the ways of approaching the asymptotics of Bell numbers is through Dobinski's expansion (2). A similar treatment can be inflicted to form (8), so that a *bona fide* expansion of the number of non–overlapping partitions can ultimately be derived.

The reader is referred to a recent work of Fédou [6] for related combinatorial models involving Bessel functions, and interesting q–analoguess. Also non–overlapping partitions are of interest in the study of some data structures in computer science. They code all possible evolutions of a stack with "inspection" or a symbol table under Knuth's model. The corresponding problems of average case *dynamic* (or "amortized") analysis [12,9,10] of data structures provided the initial motivation for this investigation of non–overlapping partitions.

2 Continued Fractions and Bessel Functions

General (set) partitions are in bijective correspondence with a class of so–called *path diagrams*. Such a correspondence produces the continued fraction expansion (3) for the ordinary generating function of Bell numbers.[2]

In essence, to a partition ϖ of $[1..n]$, we associate a *path* in the integer lattice as follows: Start from $(0,0)$. Scan the integers j from 1 to n. Move by $\vec{a} = (+1,+1)$ when j is the minimal element of a non–singleton block in ϖ; move by $\vec{d} = (+1,-1)$ when j is the minimal element of a non–singleton block in ϖ; move by $\vec{\ell} = (+1,0)$ otherwise —in this last case, we encountered either an intermediate element of a block, or a singleton element.

In this way we encode non uniquely a partition by a path formed with *ascents* ($\vec{a}$), *descents* ($\vec{d}$) and *level* ($\vec{\ell}$) steps.

A complete encoding of an unconstrained partition ϖ is obtained by supplementing a numerical sequence which connects intermediate and maximal elements to their respective classes, ordered for instance by age rank. (For this purpose, singletons will be treated as intermediate elements.) Thus when scanning j, if h blocks are open then a descent has h possibilities, and a level step has $(h+1)$ possibilities—one more, because of singletons. The pair formed with the path and its number sequence determines the partition and is called a path diagram.

It is now easy to see, on associated path diagrams, the rule defining NOP's: If an element is maximal in its block, it has to close the most recently opened class. In this way, the number of possibilities for a descent is reduced to 1, while the number of possibilities for a level step remains equal to $(h+1)$. For the non–overlapping partition that we considered earlier (4), the encoding is

$$\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ a & a & \ell & a & \ell & \ell & d & d & d & \ell & a & \ell & d \\ - & - & 0 & - & 2 & 1 & - & - & - & 0 & - & 1 & - \end{array}$$

(Age ranks are numbered from older to younger, starting from 0.)

From the combinatorial theory (Thm. 1 and Prop. 7 of [7]), the generating function of a class of path diagrams admits a continued fraction expansion, where the number of possibilities for a level step appears in the

[2] The reader can either accept Proposition 1 as a starting point for our asymptotic treatment or else refer to [11,7] for detailed definitions of path diagrams and background information on combinatorial aspects of continued fractions.

denominators while the possibilities for ascents and descents appear in the numerators. It follows that the change in the possibility rule for path diagrams (from unconstrained partitions to non–overlapping partitions) is exactly reflected by the change from (3) to (5).

Proposition 1 *The ordinary generating function of non–overlapping partitions, $B^*(z) \equiv \sum_{n\geq 0} B_n^* z^n$ admits the formal continued fraction expansion:*

$$B^*(z) = \cfrac{1}{1 - 1\cdot z - \cfrac{z^2}{1 - 2\cdot z - \cfrac{z^2}{1 - 3\cdot z - \cfrac{z^2}{\cdots}}}}. \tag{9}$$

We now move to the world of Bessel functions, of which however we shall only use the most basic properties. From their defining equation (6), the fundamental recurrence follows:

$$J_{\nu+1}(x) = 2\nu x^{-1} J_\nu(x) - J_{\nu-1}(x) \tag{10}$$

To prime the continued fraction pump, rewrite this relation as

$$\frac{J_{\nu-1}(x)}{J_\nu(x)} = 2\nu x^{-1} - \frac{1}{J_\nu(x)/J_{\nu+1}(x)},$$

and simply iterate

$$\frac{J_{\nu-1}(x)}{J_\nu(x)} = 2\nu x^{-1} - \cfrac{1}{2(\nu+1)x^{-1} - \cfrac{1}{2(\nu+2)x^{-1} - \cfrac{1}{J_{\nu+2}(x)/J_{\nu+3}(x)}}}.$$

If we repeat the process *ad infinitum*, and further substitute $x = 2$, we get

$$\frac{J_{\nu-1}(2)}{J_\nu(2)} = \nu - \cfrac{1}{\nu + 1 - \cfrac{1}{\nu + 2 - \cfrac{1}{\nu + 3 - \cfrac{1}{\cdots}}}}, \tag{11}$$

whose shape closely resembles (9). The formal derivation above is also valid analytically, as was shown by Hurwitz (cf [15, Sec. 9.65]), and Eq. (11) remains valid for all complex ν.

The continued fraction expansion of Eq. (11) admits itself an asymptotic expansion in descending powers of ν. The connection between (11) and (9) is achieved by the correspondence $\nu \leftrightarrow -z^{-1}$. We have thus obtained:

Proposition 2 *The asymptotic expansion, as $\nu \to +\infty$, of a quotient of consecutive Bessel functions is expressed using Bessel numbers by*

$$\frac{J_{\nu-1}(2)}{\nu J_\nu(2)} \sim 1 - \sum_{n\geq 0} B_n^* \frac{(-1)^n}{\nu^{n+2}} \tag{12}$$

Thus, the quotient $J_{\nu-1}(2)/(\nu J_\nu(2))$ plays the rôle of an *ordinary generating function* of the Bessel numbers B_n^*.

3 The Mittag–Leffler Expansion

For ease of notation, we define

$$j(\nu) = J_\nu(2) \qquad \text{and} \qquad h(\nu) = \frac{J_{\nu-1}(2)}{\nu J_\nu(2)}. \tag{13}$$

We propose to investigate[3] first the geometry of zeroes of function $j(\nu)$ which provides the poles of $h(\nu)$. By a local analysis, we determine the simple elements that compose $h(\nu)$. Putting these elements together yields the partial fraction expansion of $h(\nu)$.

The geometry of the zeroes of $j(\nu)$ is amazingly regular. From numerical computations, we find that the first few zeroes (with modulus at most 10) are negative reals that are extremely well approximated by negative integers.

$\zeta_1 = -0.25380\ 58170$, $\zeta_2 = -1.78932\ 13526$, $\zeta_3 = -2.96105\ 88806$,
$\zeta_4 = -3.99604\ 79973$, $\zeta_5 = -4.99977\ 43198$, $\zeta_6 = -5.99999\ 18413$,
$\zeta_7 = -6.99999\ 97949$, $\zeta_8 = -7.99999\ 99961$,
$\zeta_9 = -8.99999\ 99999\ 45511$, $\zeta_{10} = -9.99999\ 99999\ 99380$.

Lemma 1 *All zeroes of function $j(\nu) \equiv J_\nu(2)$ are negative real numbers. The r–th negative zero, ζ_r, of $j(\nu)$ satisfies*

$$\zeta_r = -r + \frac{1}{r!\,(r-1)!} + O(\frac{1}{(r!)^2}). \tag{14}$$

[3] Most results in this and the next section derive from a few simple key observations followed by trite real analysis. We shall thus limit ourselves to indicating the main steps in the proofs.

This lemma is a quantitative version of a result of Coulomb who first observed that the zeroes of $J_\nu(x)$, with x fixed, are asymptotic to the negative integers [3].

Proof. We start from the equation defining $j(\nu)$:

$$\begin{aligned} j(\nu) &= \frac{1}{\Gamma(\nu+1)} - \frac{1}{1!\,\Gamma(\nu+2)} + \frac{1}{2!\,\Gamma(\nu+3)} \cdots \\ &= \frac{1}{\Gamma(\nu+1)}\left[1 - \frac{1}{1!\,(\nu+1)} + \frac{1}{2!\,(\nu+1)(\nu+2)} - \cdots\right]. \end{aligned} \tag{15}$$

From the first form, no negative integer can be a root of $j(\nu)$. The basic observation for the proof is the following: When $\nu \approx -r$ ($r \in \mathbf{N}$), cancellation of the series expansion of $j(\nu)$ comes predominantly from cancellation of two terms

$$1 + \frac{(-1)^r}{r!\,(\nu+1)(\nu+2)\cdots(\nu+r)},$$

and we denote this expression by $f(\nu)$. We may also freely assume that $r > 10$ since zeroes of modulus less than 10 have been characterized.

A. *The approximate equation.* We propose to analyze the zero of $f(\nu)$ which lies in the vicinity of $-r$ and let ζ_r^* denote that zero. Set $\zeta_r^* = -r + \epsilon$, with $\epsilon = o(1)$ as $r \to \infty$. Using the complement formula for the Gamma function, $f(\nu)$ transforms into

$$f(\nu) = 1 - \frac{(-1)^r}{r!} \frac{\pi}{\sin \pi\nu\, \Gamma(-\nu)\Gamma(\nu+r+1)},$$

so that ϵ is a root of

$$\begin{aligned} \frac{\sin \pi\epsilon}{\pi} &= \frac{1}{r!\,\Gamma(1+\epsilon)\Gamma(r-\epsilon)} \\ &= \frac{1}{r!\,(r-1)!}(1 + O(\epsilon r)). \end{aligned}$$

From this relation, we find that ϵ —hence ζ_r^*— exists and ζ_r^* satisfies

$$\zeta_r^* = -r + \frac{1}{r!\,(r-1)!} + O(\frac{1}{(r!)^2}).$$

B. *The exact equation.* Considering again function $f(\nu)$ and noting that the other terms that compose the expansion of $j(\nu)$ are small, we find that

$j(\nu)$ has a real root in the interval $[-r-2\eta, -r+2\eta]$, where $\eta = 1/(r!\,(r-1)!)$. Introduce now the complementary function

$$g(\nu) = \sum_{m\neq 0,r} \frac{(-1)^m}{m!\,(\nu+1)(\nu+2)\cdots(\nu+m)},$$

so that $f(\nu)+g(\nu) = \Gamma(\nu+1)j(\nu)$. Elementary asymptotic expansions show that on the circle centered at $-r$ and with radius 2η, we have $|g(\nu)| < |f(\nu)|$. Thus, by Rouché's theorem, $f(\nu)$ and $j(\nu)$ have the same number of zeroes inside this circle.

The possible zeroes of $j(\nu)$ around the negative integers have been thus been localized. Clearly $g(\nu) = -\nu^{-1} + O(\nu^{-2})$ as $\nu \to -\infty$. A simple modification of the argument in Part A then reveals that

$$|\zeta_r - \zeta_r^*| = O(\frac{1}{(r!)^2}).$$

C. *There are no other zeroes.* A detailed proof is given in [3], so that we just summarize the essence of the argument in our context for completeness. The only way that $j(\nu)\Gamma(\nu+1)$ can be zero is if one of the terms in the expansion becomes large enough to cancel the leading term, 1, in (15). This can only happen if one of the quantities $(\nu+1), (\nu+2), \ldots$ becomes small, i.e. if ν is close enough to a negative real number; but zeroes in these regions have already been characterized by the argument of part B. ∎

Lemma 2 *When $\nu \to \zeta_r$, we have*

$$h(\nu) \sim \frac{c_r}{\nu - \zeta_r} \qquad \textit{where} \qquad c_r = \frac{j(\zeta_r - 1)}{\zeta_r j'(\zeta_r)}. \tag{16}$$

Furthermore, the coefficients c_r satisfy for $r \to +\infty$:

$$c_r = \frac{-1}{(r-1)!\,r!}\,(1 + O(\frac{1}{r})). \tag{17}$$

Proof. The first part is obvious. The second part follows from the values of ζ_r, the series definition of Bessel functions (15), and elementary growth properties of the Gamma function.

Consider $j(-r)$ and $j'(-r)$. The first r terms of the expansion of $j(-r)$ reduce to 0, so that $j(-r) = (-1)^r/r! + (-1)^{r+1}/(1!\,(r+1)!) + \cdots$. Since the residue of $\Gamma(s)$ at $s = -m$ (m a positive integer) is equal to $(-1)^m/m!$, we find that $j'(-r)$ is driven by its first terms, $j'(-r) = (-1)^{r-1}(r-1)! - \cdots$. In summary,

$$j(-r) = \frac{(-1)^r}{r!}(1 + O(\frac{1}{r})) \quad \text{and} \quad j'(-r) = (-1)^{r-1}(r-1)!(1 + O(\frac{1}{r})). \tag{18}$$

Similar calculations reveal that since ζ_r is very close to $-r$, then $j'(\zeta_r)$ is closely approximated by $j'(-r)$, and actually $j'(\nu)$ is fairly stationary around negative integers. Once this has been established, the asymptotic value of $j(\zeta_r - 1)$ derives from the near equality

$$j(\zeta_r - 1) - j(\zeta_{r+1}) \approx (\zeta_r - 1 - \zeta_{r+1})j'(-r-1).$$

In this way, we gather the estimates

$$j(\zeta_r - 1) = \frac{(-1)^{r+1}}{(r+1)!}(1 + O(\frac{1}{r})) \quad \text{and} \quad j'(\zeta_r) = (-1)^{r-1}(r-1)!\,(1 + O(\frac{1}{r})). \tag{19}$$

From there, the asymptotic form of c_r follows. ■

It only remains to collect elements representing the local behaviour of $h(\nu)$ around its poles. To take care of the pole at 0, define $c_0 = j(-1)/j(0)$ and $\zeta_0 = 0$. Consider next the sum

$$h^*(\nu) = \sum_{r \geq 0} \frac{c_r}{\nu - \zeta_r}. \tag{20}$$

Due to the fast decrease of coefficients c_r, $h^*(\nu)$ is well defined and meromorphic for all ν. Thus the function

$$d(\nu) = h(\nu) - h^*(\nu)$$

is an entire function of ν.

We observe that $h^*(\nu)$ tends to 0 along large circles of radius $R = r + \frac{1}{2}$ with $r \in \mathbf{N}$. On such circles, from the series defining Bessel functions, we see that $j(\nu-1)/(\nu j(\nu))$ tends to 1. Thus $d(\nu)$ is an entire function that is bounded on a family of arbitrarily large circles centered at the origin. Therefore, by Liouville's theorem, $d(\nu)$ is a constant (actually, $d(\nu) = d(+\infty) = 1$). We have thus established:

Lemma 3 *The function $h(\nu)$ admits the Mittag–Leffler expansion*

$$h(\nu) \equiv \frac{J_{\nu-1}(2)}{\nu J_\nu(2)} = 1 + \sum_{r \geq 0} \frac{c_r}{\nu - \zeta_r}, \tag{21}$$

where

$$c_r = \frac{-1}{(r-1)!\, r!}(1+O(\frac{1}{r})) \quad \textit{and} \quad \zeta_r = -r + \frac{1}{r!\,(r-1)!} + O(\frac{1}{(r!)^2}). \qquad (22)$$

The existence of a Mittag–Leffler expansion was first derived by Maki [14, Th. 5.1] as a consequence of general considerations on orthogonal polynomials. Our lemma thus also constitutes a quantitative version of Maki's result.

4 Asymptotic Forms

The Mittag–Leffler expansion (21) yields an asymptotic expansion of $h(\nu)$ as $\nu \to +\infty$. To see it, start with the identity

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots u^{n+1} + \frac{u^{n+2}}{1-u},$$

so that $(r > 0)$

$$\frac{1}{\nu - \zeta_r} = \frac{1}{\nu}\frac{1}{1-\frac{\zeta_r}{\nu}} = \sum_{m=0}^{n+1} \frac{\zeta_r^m}{\nu^{m+1}} + \frac{\zeta_r^{n+2}}{\nu^{n+2}(\nu - \zeta_r)}.$$

Combining these expansions leads to the identity

$$h(\nu) = 1 + \frac{c_0}{\nu} + \sum_{m=0}^{n+1}\left(\sum_{r=1}^{\infty} \frac{c_r\zeta_r^m}{\nu^{m+1}}\right) + \sum_{r=0}^{\infty} \frac{c_r\zeta_r^{n+2}}{\nu^{n+2}(\nu-\zeta_r)}.$$

The last sum is clearly $O(\nu^{-n-3})$ as $\nu \to +\infty$. This provides an exact form for the coefficients in the asymptotic expansion of $h(\nu)$, which by (12) are the Bessel numbers.

Proposition 3 *Bessel numbers are expressible in terms of the zeroes ζ_r of Bessel function $j(\nu)$ by*

$$B_n^* = -\sum_{r\geq 1} c_r\, |\zeta_r|^{n+1} \qquad \textit{with} \qquad c_r = \frac{j(\zeta_r - 1)}{\zeta_r j'(\zeta_r)}. \qquad (23)$$

By Lemmas 1, 2, and 3, all quantities entering Eq. (23) have known asymptotic forms, whence:

Theorem 1 *Bessel numbers satisfy asymptotically*

$$B_n^* \sim \sum_{k\geq 1} \frac{k^{n+2}}{(k!)^2}. \tag{24}$$

Proof. Lemma 1 provides a very accurate expression for ζ_r. Lemma 2 contains an estimate of coefficients c_r, where the relative error term in approximation (19) is $O(r^{-1})$. Thus, we find

$$B_n^* = \sum_{k\geq 1} \frac{k^n}{((k-1)!)^2} + \sum_{k\geq 1} O(\frac{1}{k})\frac{k^n}{((k-1)!)^2}.$$

Letting S_n denote the sum appearing in (24), we thus have $B_n^* = S_n + O(S_{n-1})$. But by our next proposition, we have $S_{n-1} = o(S_n)$, and the result follows. ■

Numerically, the error of approximation (24) is 22% when $n = 50$ and 10% when $n = 100$. A more orthodox asymptotic form can also be produced. Apart from subexponential factors, Bessel numbers grow like

$$B_n^* \approx \left(\frac{n}{2e\log n}\right)^n.$$

Proposition 4 *Bessel numbers have the asymptotic form*

$$B_n^* \sim \frac{1}{\sqrt{2\pi n}} \frac{\omega^{n+3}}{(\omega!)^2}, \tag{25}$$

where $\omega \sim n/(2\log n)$ is the positive root of equation $n+2 = 2\omega\log\omega$.

Proof. We only need to estimate the sum S_n appearing in the right hand side of Eq. (24). In passing, we shall also check that $S_{n-1} = o(S_n)$. The proof follows closely the asymptotic analysis of Bell numbers using the Laplace method for sums as detailed in De Bruijn's book [4], so that we need only indicate the main steps.

By Stirling's formula, the general term in the sum (24) roughly equals

$$(2\pi k)^{-1}\exp(t(k)), \qquad \text{where} \qquad t(k) = (n+2)\log k - 2k\log k + 2k.$$

By cancelling $t'(k)$, we find that the index $k_{\max}$ of the largest term in S_n is close to ω, where ω satisfies $n+2 = 2\omega\log\omega$. Observe that $\omega \sim n/(2\log n)$.

The second derivative $t''(k)$ is $-(n+2)k^{-2} - 2k^{-1}$, so that an interval $|k - k_{\max}| < n^{1/2}$ provides the dominant contribution to the sum.

In this way, we find for S_n the approximate form

$$\begin{aligned} S_n &\sim \frac{\exp(t(\omega))}{2\pi\omega} \sum_k \exp\,(\frac{1}{2} t''(\omega)(k-\omega)^2) \\ &\sim \frac{\exp(t(\omega))}{2\pi\omega} \int_{-\infty}^{+\infty} \exp\,(\frac{1}{2} t''(\omega)\kappa^2)\, d\kappa \\ &\sim \frac{\exp(t(\omega))}{2\pi\omega} \sqrt{\frac{2\pi}{|t''(\omega)|}}. \end{aligned}$$

This concludes the proof of the theorem. ∎

Notice also that a full asymptotic expansion of $\omega = \omega(n)$ can be obtained and then plugged into (25). In this way, we get

$$\frac{1}{n} \log B_n^* = \log n - \log\log n - \log(2e) + O(\frac{\log\log n}{\log n}).$$

5 Conclusions

Some of the classical developments relative to Bell numbers have parallels for Bessel numbers that we now briefly indicate.

Function $h(\nu)$ plays the rôle of an ordinary generating function for Bessel numbers. We may observe that $h(\nu)$ is for Bessel numbers the counterpart of the incomplete Gamma function in the world of Bell numbers: for Bell numbers, we have

$$\frac{1}{e} \int_0^1 u^{t-1} e^u \, du = \frac{1}{e} \sum_{r \geq 0} \frac{1}{r!\,(z+r)} \sim \sum_{n \geq 0} B_n \frac{(-1)^n}{z^{n+1}}, \tag{26}$$

where the poles are now *exactly* at the negative integers.

Other parallels are relative to refined counting by number of classes (Stirling numbers), elimination of singleton classes ("2–associated numbers"), convergents to continued fractions that are connected to orthogonal polynomials, as well as auxiliary properties of continued fraction expansions (Hankel determinants and congruence properties).

1. Let $S_{n,k}^*$ represent the number of non–overlapping partitions of an n–set into k equivalence classes. These numbers are analogues of the Stirling number of the first kind. A refinement of the argument of Section 2 provides

for their bivariate generating function a continued fraction expansion

$$\sum_{n,k\geq 0} S^*_{n,k} u^k z^n = \cfrac{1}{1 - u\cdot z - \cfrac{u\cdot z^2}{1-(1+u)\cdot z - \cfrac{u\cdot z^2}{1-(2+u)\cdot z - \cfrac{u\cdot z^2}{\cdots}}}}. \tag{27}$$

2. Convergents of the continued fraction of Bell numbers involve Poisson–Charlier polynomials. If we consider the h–th convergent

$$K^{[h]}(\nu) = \cfrac{1}{\nu+1-\cfrac{1}{\nu+2-\cfrac{1}{\ddots\,\cfrac{}{\nu+h}}}}, \tag{28}$$

to fraction (11), we find instead a form,

$$K^{[h]}(\nu) = \frac{r_{h-1}(\nu+2)}{r_h(\nu+1)},$$

involving (modified) Lommel polynomials [15, pp.296–303], [1, p.188],

$$r_m(\nu) = \sum_{n=0}^{\lfloor m/2\rfloor} \binom{m-n}{n} (-1)^n (\nu+n)^{\underline{m-2n}}, \ \text{ with } x^{\underline{a}} = x(x+1)\cdots(x+a-1).$$

As follows from the combinatorial theory of continued fractions [7], convergents K are related to enumeration of NOP's of bounded "height", height in a NOP being defined as the maximum number of block supports covering an element of $[1..n]$. (For instance, the partitions of Figure 1 both have height equal to 3).

3. If we forbid partitions to contain singleton classes, we obtain what can be called, after Comtet, 2–associated numbers [2, p.221]. Let ${}_2B^*_n$ denote the number of NOP's of size n without singletons. A modified form of our basic continued fraction (5) expresses the ordinary generating function of the ${}_2B^*_n$ as

$$\cfrac{1}{1-0\cdot z-\cfrac{1\cdot z^2}{1-1\cdot z-\cfrac{1\cdot z^2}{1-2\cdot z-\cfrac{1\cdot z^2}{\cdots}}}}. \tag{29}$$

From (11), we see that now $J_\nu(2)/J_{\nu-1}(2)$ acts as an ordinary generating function for these numbers, and the asymptotic analysis can be developed accordingly. It is 2–associated Bessel numbers that are most directly relevant to computer science applications.

4. Values of Hankel determinants are known to be related to coefficients of continued fraction expansions. For instance, we have here

$$\begin{vmatrix} B_0^* & B_1^* & \cdots & B_n^* \\ B_1^* & B_2^* & \cdots & B_{n+1}^* \\ \vdots & \vdots & \ddots & \vdots \\ B_n^* & B_{n+1}^* & \cdots & B_{2n}^* \end{vmatrix} = 1.$$

5. Congruence properties modulo prime numbers also derive from such expansions [8]: In contrast to Bell numbers, Bessel numbers are *not* eventually periodic modulo any prime. For instance, setting $f_n \equiv B_n^* \bmod 2$, we find

$$f_0 = 1, \qquad f_{2n+1} = 1, \qquad f_{2n+2} = 1 - f_n.$$

In other words, the sequence is 2–automatic in the sense of [5].

6. Dynamic analysis of data structures requires investigating the distribution of "altitudes" in a random NOP. Analytically, this requires finding the speed of convergence of zeroes of Lommel polynomials to zeroes of Bessel functions. The authors plan to examine this problem in a companion paper.

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