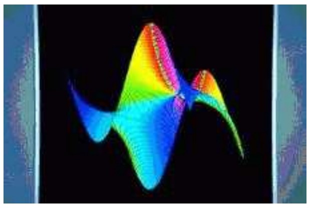
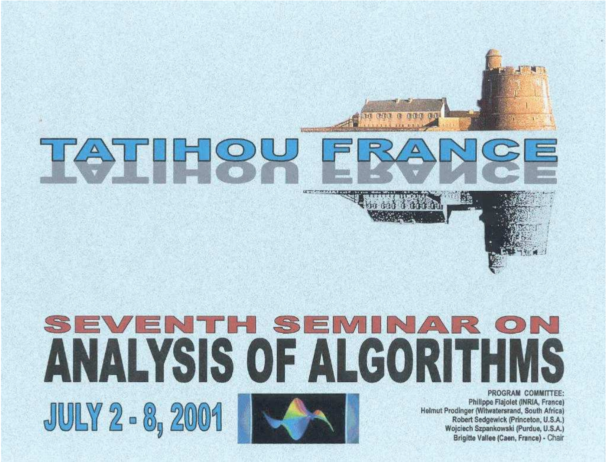




Algebraic Analytic Asymptotics

Philippe Flajolet, INRIA

The Seventh Seminar on Analysis of Algorithms

$\Longrightarrow$ For COMBINATORIAL APPLICATIONS:

- ALGEBRAIC FUNCTIONS

$$P(z, Y) = 0$$

- ASYMPTOTICS OF COEFFICIENTS

$$\text{coeff. } [z^n]Y(z) \overset{?}{\approx} CA^n n^{p/q}$$

- by ANALYTIC METHODS, i.e., SINGULARITIES

Based on Fl-Sedgewick, 01/2001,
`http:algo.inria.fr/flajolet/Publications`
& B. Salvy, C. Chabaud.

I. Algebraic functions: what they are.

II. Algebraic models of combinatorics:

- Context–free languages and specifications
- Trees
- Walks and kernel method
- Maps and quadratic method
- $\cdots$

III. Coefficient asymptotics is essentially decidable!

- General algebraic functions
- Positive case
- Special systems: Drmota-Lalley-Woods' Theorem

IV. Applications

I. Algebraic functions

What are they?

Two equivalent definitions:

- As solutions over C of

$$P(z, Y) = 0$$

  $=$ an *algebraic curve*.

- As components of systems of equations

$$y_1 \ : \qquad \begin{cases} y_1 & = \Phi_1(z; y_1, \ldots, y_r) \\ \vdots & \vdots \qquad\qquad\qquad\qquad \vdots \\ y_r & = \Phi_1(z; y_1, \ldots, y_r) \end{cases}$$

  $=$ projection of an *algebraic variety*.

Theorem. Elimination is always possible:

equation $\Leftrightarrow$ system

Proofs: Nonconstructive: Ideals;

Constructive: Resultants;

Constructive$'$: Groebner bases.

Consider an algebraic curve

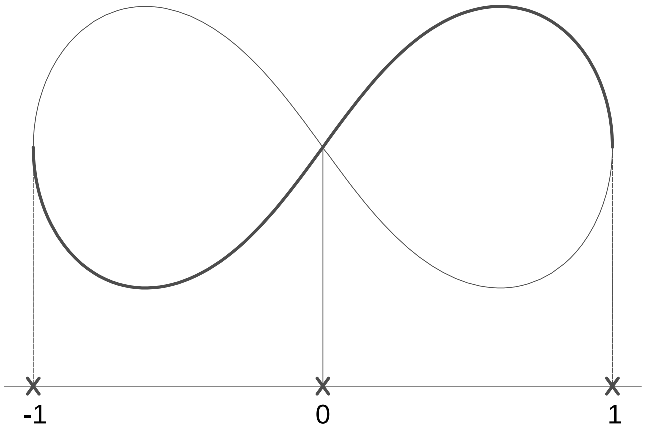
$$P(z, y) = 0$$

with $d := \deg_y P(z, y)$.

There are "usually" d distinct roots (finitely many exceptions) that organize themselves into **branches** of an analytic function (by Implicit Functions).

Exceptional points are multiple points and branch points.

Lemniscate: $(z^2 + y^2)^2 - (z^2 - y^2) = 0$.



EXAMPLE. The Catalan curve

$$\mathcal{B} = [\,] + \{\odot\} \times \mathcal{B} \times \mathcal{B}$$

$$B(z) = 1 + zB(z)^2$$

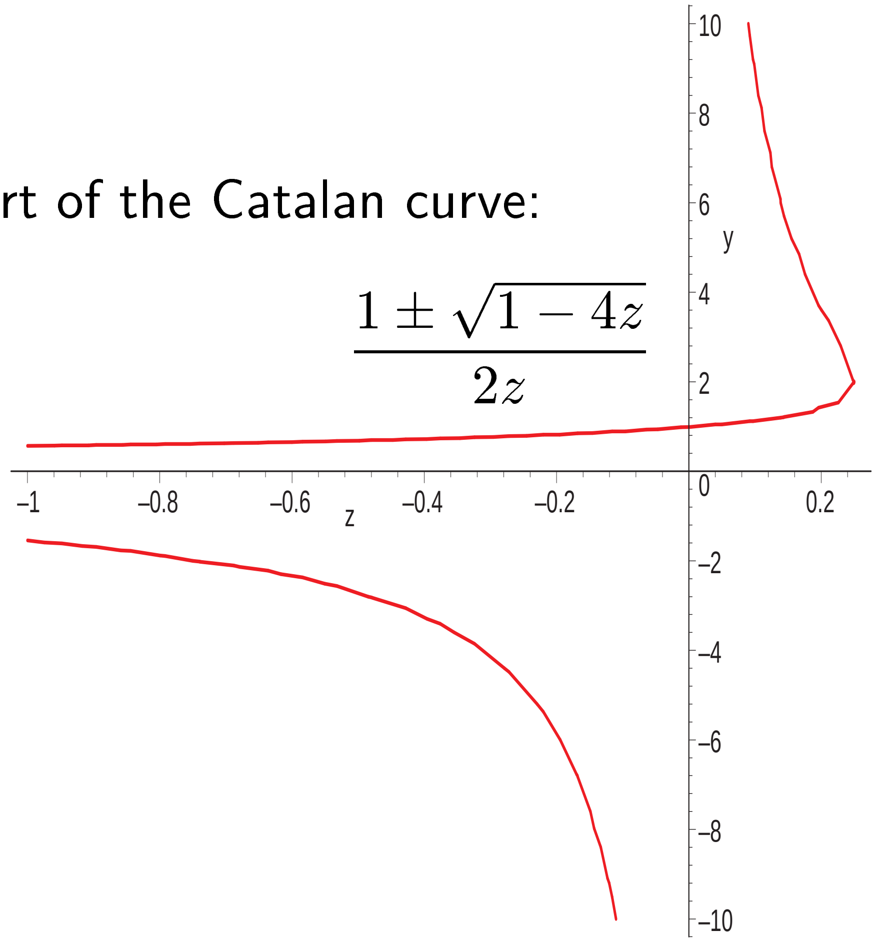
The two branches over the reals are

$$Y_1 = \frac{1 - \sqrt{1-4z}}{2z}, \quad Y_2 = \frac{1 - \sqrt{1-4z}}{2z}$$

And $B(z)$ coincides with Y_1 near 0.

Real part of the Catalan curve:

$$\frac{1 \pm \sqrt{1-4z}}{2z}$$



Combinatorics dictates initial conditions

$$B(z) = 1 + 1z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

CONNECTION PROBLEM:

Which branch to choose?

Where does it lead to?

II. Algebraic Models of Combinatorics

- context-free languages and specifications
- trees
- wlaks, excursions, . . .
- maps

The SYMBOLIC DICTIONARY

From constructions to equations over GF's

Combinatorial construction	Generating Functions
$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$A(z) = B(z) + C(z)$
$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(z) = B(z) \cdot C(z)$
$\mathcal{A} = \mathfrak{Seq}\ \mathcal{B}$	$A(z) = \dfrac{1}{1 - C(z)}$

$\mathfrak{Seq}$ is definable rec.: $\mathcal{A} = \mathbf{1} + \mathcal{A} \times \mathcal{C}$.

Definition. A context-free specification involves unions and products only, possibly in recursive fashion.

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E.g. Catalan domain: Bin Trees, Gen Trees, Triangulations

In general defines a family of tree-like objects
$=$ Trees with various types of nodes.

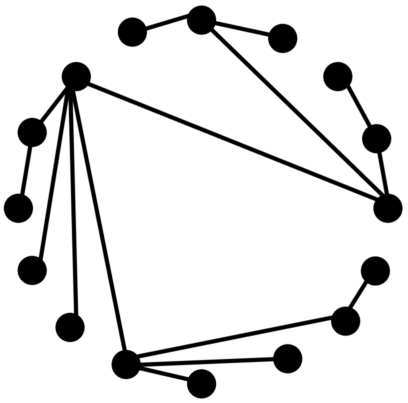
Definition. A contex-free language is the set of *words* obtained as leaf labels in preorder traversal.

Unambiguous: unique reconstruction: word $\leadsto$ tree.

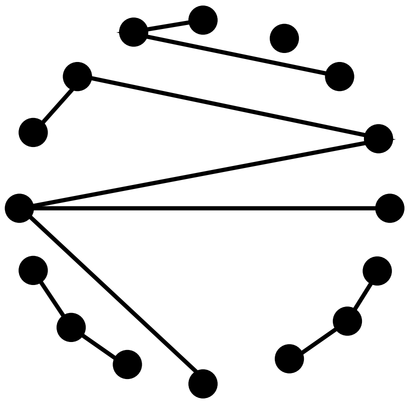
Dyck paths, tree codes, parenthesis systems, generalized Dyck paths and walks (Banderier,...)

Theorem. The OGF of a context-free class or an unambiguous context-free language is an algebraic function.
A system of polynomials equations is obtained by the symbolic dictionary.

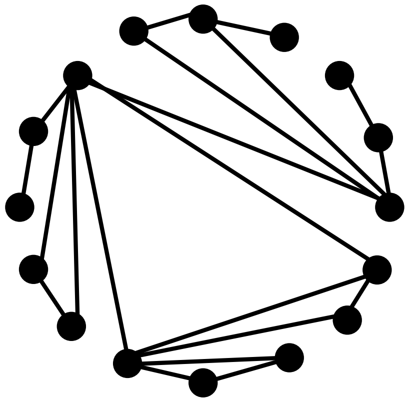
Example. **Noncrossing graphs** [F.Noy, Disc. Math'99]



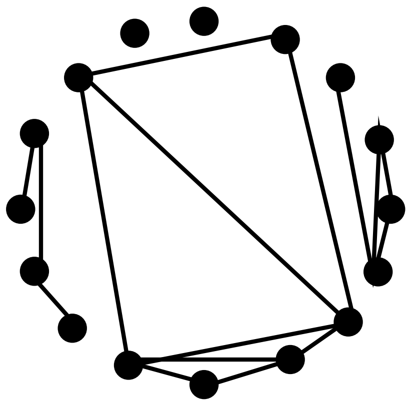
(tree)



(forest)



(connected graph)



(graph)

Configuration / OGF	Coefficients (exact / asymptotic)
Trees (*EIS*: **A001764**)	$z + z^2 + 3z^3 + 12z^4 + 55z^5 + \cdots$
$T^3 - zT + z^2 = 0$	$\frac{1}{2n-1}\binom{3n-3}{n-1}$
	$\sim \frac{\sqrt{3}}{27\sqrt{\pi n^3}}(\frac{27}{4})^n$
Forests (*EIS*: **A054727**)	$1 + z + 2z^2 + 7z^3 + 33z^4 + 181z^5 \cdots$
$F^3 + (z^2 - z - 3)F^2 + (z+3)F - 1 = 0$	$\sum_{j=1}^{n} \frac{1}{2n-j}\binom{n}{j-1}\binom{3n-2j-1}{n-j}$
	$\sim \frac{0.07465}{\sqrt{\pi n^3}}(8.22469)^n$
Connected graphs (*EIS*: **A007297**)	$z + z^2 + 4z^3 + 23z^4 + 156z^5 + \cdots$
$C^3 + C^2 - 3zC + 2z^2 = 0$	$\frac{1}{n-1}\sum_{j=n-1}^{2n-3}\binom{3n-3}{n+j}\binom{j-1}{j-n+1}$
	$\sim \frac{2\sqrt{6}-3\sqrt{2}}{18\sqrt{\pi n^3}}\left(6\sqrt{3}\right)^n$
Graphs (*EIS*: **A054726**)	$1 + z + 2z^2 + 8z^3 + 48z^4 + 352z^5 + \cdots$
$G^2 + (2z^2 - 3z - 2)G + 3z + 1 = 0$	$\frac{1}{n}\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\binom{2n-2-j}{n-1-j}2^{n-1-j}$
	$\sim \frac{\sqrt{140-99\sqrt{2}}}{4\sqrt{\pi n^3}}\left(6+4\sqrt{2}\right)^n$

Dyck words: parenthesis systems

a='('; b=')'

a a a b a b b a b b a b

Cf. Stein-Waterman 1979, DNA secondary structures.

$$\mathcal{D} = \varepsilon + \nearrow \underline{\mathcal{D}} \searrow \mathcal{D}$$

Trees

Ω: $=$ allowed set of node degrees.

$$\phi(u) := \sum_{\omega \in \Omega} u^{\omega}$$

$$T(z) \ = \ z\phi(T(z))$$

Lagrangean framework for T

$$[z^n]T(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n.$$

Ω finite $\Longrightarrow$ T is algebraic.

Singularities give useful information on parameters.

Walks, bridges, meanders, excursions

[F.Banderier-01]

$\mathcal{S}$ a finite set of allowed $\pm$ steps

$$P(u) = \sum_{s \in \mathcal{S}} u^s.$$

Characteristic equation is

$$1 - zP(u) = 0.$$

Theorem. GF's of W, B, M, E are algebraic. and involve rationally branches of characetrsitic equation.

Meanders (z: length; u: final altitude)

$$F(z,u) = 1 + zP(u)F(z,u) - \{u^{<0}\}\Big(zP(u)F(z,u)\Big).$$

Kernel method solves a functional equations

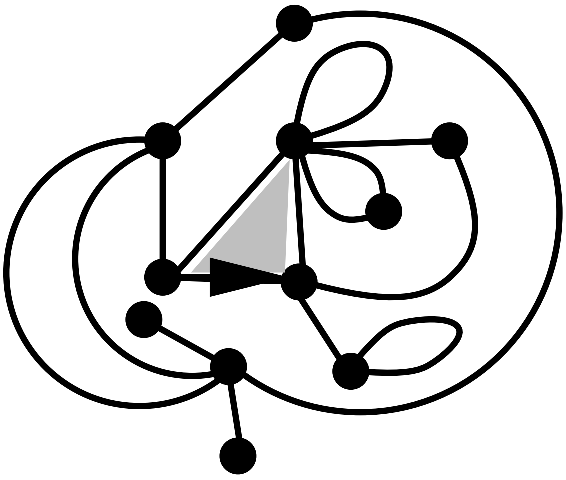
$$F(z,u)\,(1 - zP(u)) = \text{Lin. Comb.}\,[F_0(z), \ldots, F_{c-1}(z)]$$

Excursions $F(z,0) = E(z)$

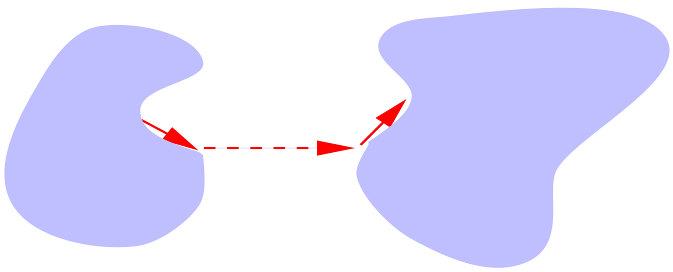
$$E(z) = \frac{(-1)^{c-1}}{z}\, u_1(z)u_2(z)\cdots u_c(z).$$

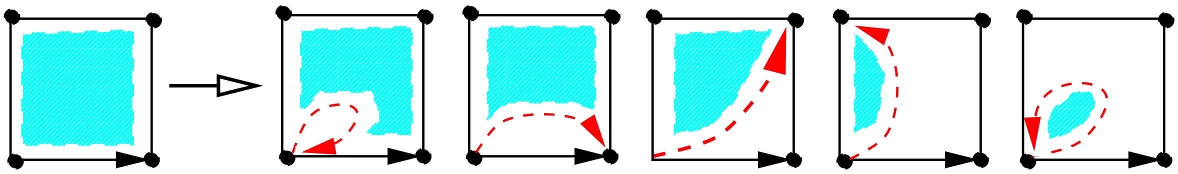
Grammars (Labelle-Yeh) don't give access to such a decomposability.

Maps



$$M(z,u) = 1 + u^2 z M(z,u)^2 + uz \frac{M(z,1) - uM(z,u)}{1-z}.$$





$$u^4 \quad \mapsto \quad zu^5 \quad + \quad zu^4 \quad + \quad zu^3 \quad + \quad zu^2 \quad + \quad zu^1.$$

Tutte's Quadratic method gives algebraic functions

Cf Schaeffer, ...

III. Coefficient Asymptotics

is decidable!

- Local analysis of singularities
- Global connections
  - General case
  - Positive functions

  Special systems: positivity + irreducibility, DLW Thm

Singularities

Darboux-Pólya method $\leadsto$ singularity analysis [FlOd90]

A DICTIONARY

$$Y(z) \sim C \cdot \left(1 - \frac{z}{\rho}\right)^{-\alpha} \implies Y_n \sim \frac{C}{\Gamma(\alpha)} \rho^{-n} n^{\alpha-1}.$$

(under "Camembert" conditions).

Coefficient extraction and asymptotic approximation commute!

$$([z^n] \approx \sigma(z)) \quad \approx \quad ([z^n]\sigma(z)).$$

Asymptotic singular scale of functions
$\implies$ Asymptotic scale of coefficients

Coefficients asymptotics —The problems

$\rightsquigarrow$ locate relevant singularities

$\rightsquigarrow$ determine exponents

Connection problem is intrinsic for algebraic functions.

Local analysis

$$P(z,y)=0, \qquad \deg_y(P)=d.$$

• Regular point: all roots are distinct; each is locally continuable into an analytic function element.

• Exceptional point: two or more roots coincide
— Multiple point: nothing happens, two$^+$ branches meet, but each is analytic; cf lemniscate
— Singularity: cannot distinguish determinations that are analytic.

At any rate: exceptional points correspond to the presence of multiple roots, hence they solve: $P=0,\ P'_y=0$, e.g.,

$$\text{Resultant}_y\left(P(z,y),\frac{\partial}{\partial y}P(z,y);\,y\right)=0$$

Includes reduction in degree: $p_d(z)=0$

Ternary trees

```
> P:=T-1-z*T^3;
```

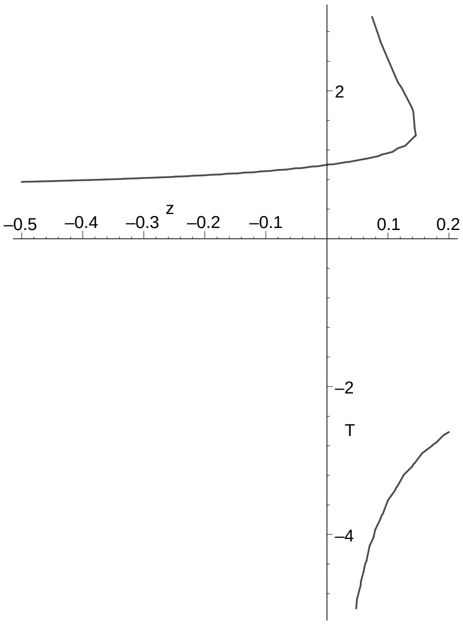
$$P := T - 1 - z\,T^3$$

```
> series(RootOf(P,T),z=0,10);
```

$$1 + z + 3\,z^2 + 12\,z^3 + 55\,z^4 + 273\,z^5 + \ldots$$

```
> resultant(P,diff(P,T),T);
```

$$-z^2\,(27\,z - 4)$$



Newton–Puiseux

At an exceptional point, there should be asymptotic cancellation: with $Z = z - z_0$ and $Y = y - y_0$:

$$\cdots + Z^a Y^b + \cdots + Z^c Y^d + \cdots = 0$$

Try

$$Y \propto Z^{p/q}$$

where

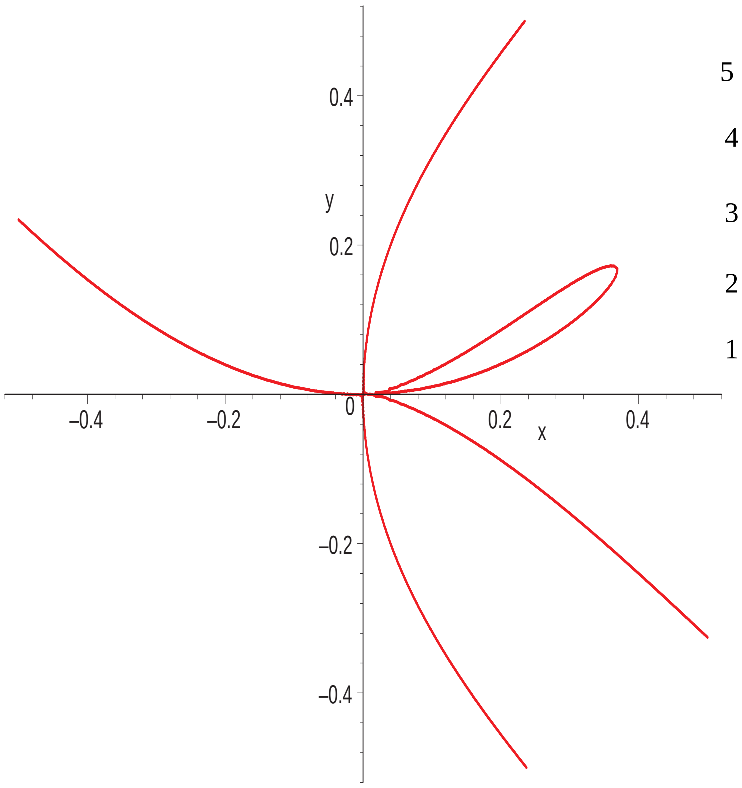
$$\frac{p}{q} a + b = \frac{p}{q} c + d, \qquad \frac{p}{q} = \frac{d-b}{a-c}.$$

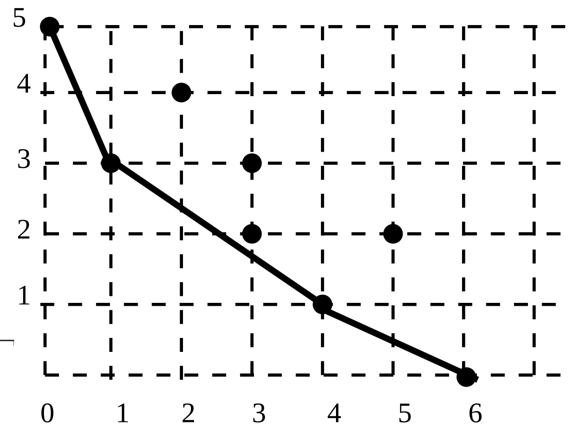
Theorem [Newton-Puiseux] At a singularity solutions group themesleves into "cycles" that can be expanded into fractional power series (Puiseux series):

$$y(z) = H((z - z_0)^{1/r}), \qquad H(w) = \sum_{j=-m}^{\infty} h_j w^j.$$

Next: Newton diagram

Newton diagram





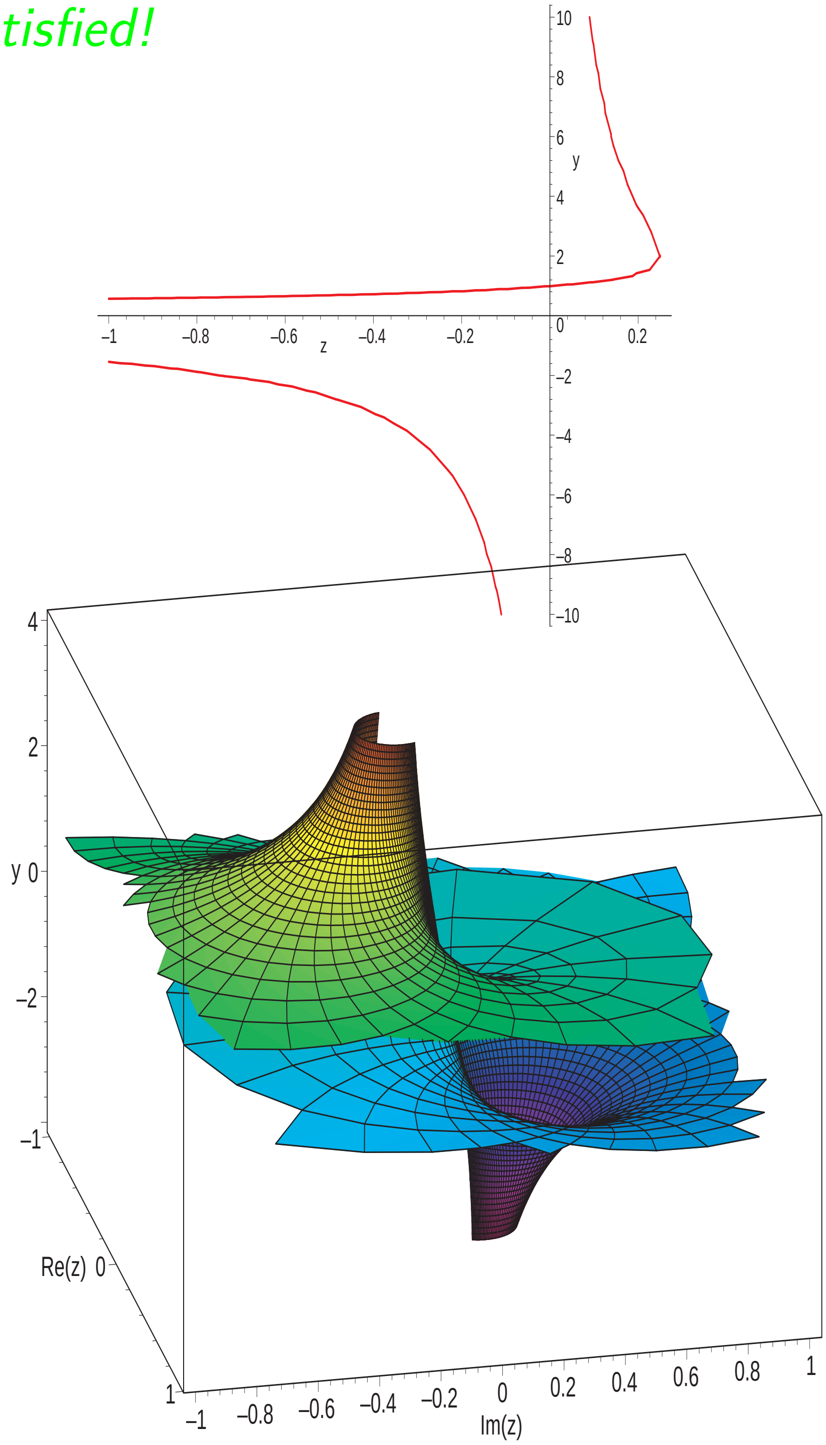
The real curve

$$P = (y - x^2)(y^2 - x)(y^2 - x^3) - x^3y^3$$

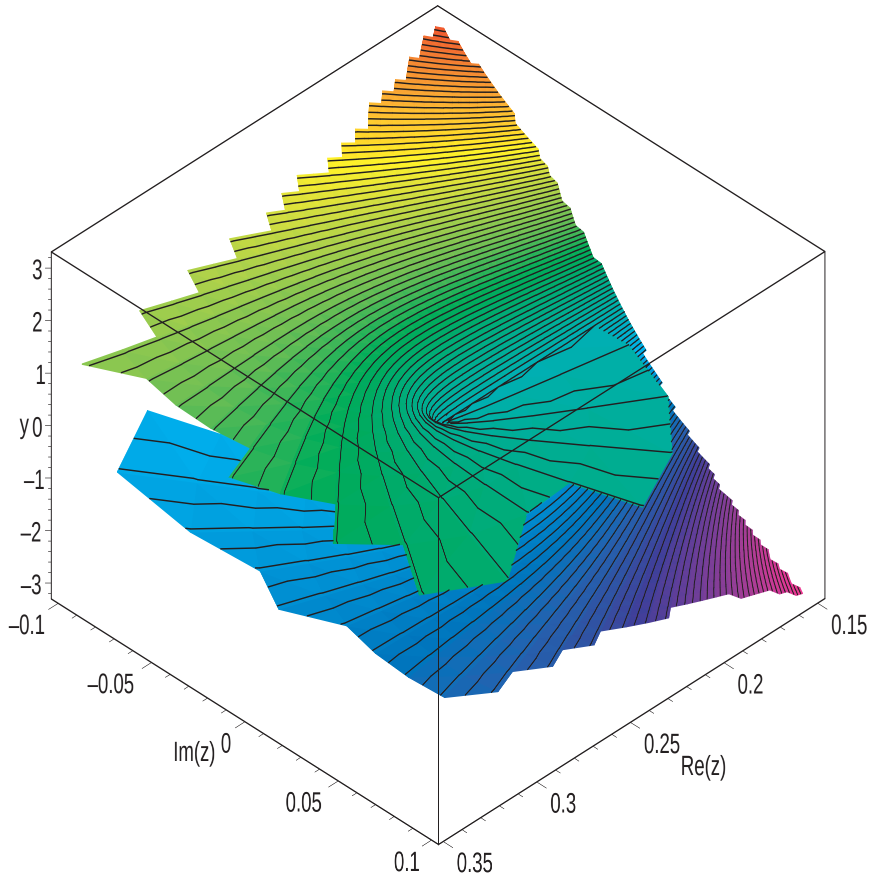
near $(0, 0)$ and its Newton diagram.

Analytically

Conditions of singularity analysis are automatically *satisfied!*



The real section of the Catalan curve (top). The complex Catalan curve with a plot of $\Im(y)$ as a function of $z = (\Re(z), \Im(z))$;



A blowup of $\Im(y)$ near the branch point at $z = \frac{1}{4}$.

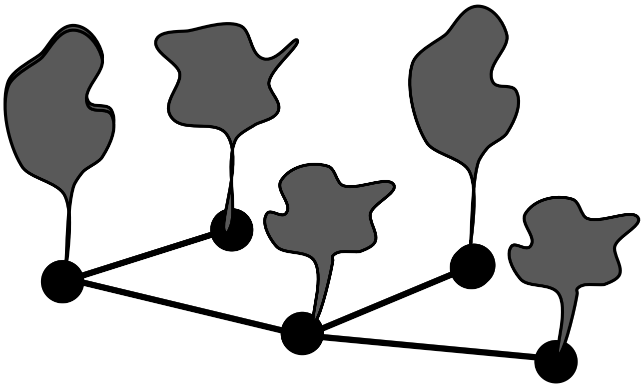
$$y(z) = c_0(1 - z/\sigma)^{p_0/q_0} + c_1(\sigma - z)^{p_1 q_1} + \cdots$$

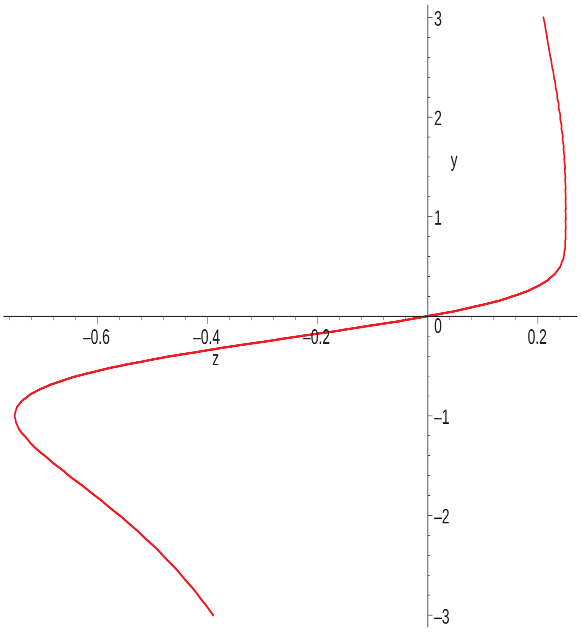
Each singular element translates into

$$\approx \frac{c_0}{\Gamma(-p_0/q_0)} \sigma^{-n} n^{-p_0/q_0 - 1}$$

EXAMPLE. Supertrees (Kemp's multiD trees?)

$$P(z, y) = zy^4 - y^3 + (2\,z + 1)\,y^2 - y + z.$$





This is "trees on trees"; $y(z) = z + z^2 + 3z^3 + 8z^4 + \cdots$;
$b(z) = (1 - \sqrt{1 - 4z^2})/2$; take $y(z) := b(zb(z))_{z^2 \mapsto z}$.

$$P(z,y) = zy^4 - y^3 + (2\,z+1)\,y^2 - y + z.$$

$$\mathcal{R} = z(4z+3)(4z-1)^3,$$

The translation $z = \frac{1}{2} - Z, y = 1 + Y$ transforms P into

$$\tilde{P}(Z,Y) = \frac{1}{4}Y^4 - ZY^4 - 4ZY^3 - 8ZY^2 - 8ZY - 4Z.$$

$$y(z) = 1 - 2\sqrt[4]{\frac{1}{4} - z} + \cdots$$

so that

$$[z^n]y(z) \sim \frac{1}{\sqrt{8}\Gamma(\frac{3}{4})} 4^n\, n^{-5/4},$$

Coefficients Asymptotics for Algebraic Functions

Theorem Let $f(z)$ be an alg. fun. (branch).
— One dominant singularity at α_1:

$$f_n \sim \alpha_1^{-n} \left(\sum_{k \geq k_0} d_k n^{-1-k/\kappa} \right),$$

where $k_0 \in \mathbb{Z}$ and κ is an integer ≥ 2. — Several: a finite linear combination of such plus exponentially smaller error terms.

Proof. Newton-Puiseux expansion + singularity analysis. Folklore. E.g. F'87, on Ambiguity and Transcendence"

Question. How to determine which exceptional points are relevant? ***Connection problem!***

Special cases: Trees, Meir and Moon; $\sqrt{\ }$-sing &c.

Excursions: F-Banderier; $\sqrt{\ }$-sing

$\leadsto$ General: Decision procedure for equations
+ modification for positive functions.

$\leadsto$ Semi-general: Positive Irreducible systems.
Drmota-Lalley-Woods' structural theorem: also $\sqrt{\ }$-singularity.

Resultants.

Given $P = a_0 x^\ell + \cdots$ and $Q = b_0 x^m + \cdots$:

$$\mathcal{R}(P, Q, x) = \det \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{\ell-1} & a_\ell \\ b_0 & b_1 & b_2 & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{m-1} & b_m \end{vmatrix}.$$

If P and Q have roots α_j and β_j:

$$\mathcal{R} = a_0^\ell b_0^m \prod_{i,j} (\alpha_i - \beta_j) = a_0^m \prod_i Q(\alpha_i) = (-1)^{\ell m} b_0^\ell \prod_j P(\beta_j).$$

$$\mathcal{R}(P(x), P'(x), x) = a_0^\ell \prod_{i \neq j} (\alpha_i - \alpha_j).$$

If P is made monic, separation distance between roots is

$$\delta := \left(\mathcal{R}(P(x), P'(x), x)\right)^{2/(n(n-1))}.$$

Algorithm ACA (Algebraic Coeff. Asympt.) $P(z, y) = 0$

1. *Preparation*. Discriminant $R = \mathcal{R}(P, P'_y, y)$;
Compute exceptional set Ξ = roots of R
Determine, by Newton-Puiseux, expansions of all branches, for $\alpha \in \Xi \cup \{0\}$.
Identify sought $Y(z)$ from initial conditions.

2. *Dominant singularities.*
Arrange Ξ in layers, $\Xi_1, \Xi_2, \ldots$ according to $|\alpha|$;
Examine each candidate $\sigma \in \Xi_1$ by
matching $Y(z)$ and the local expansion y_σ at a point between 0 and σ; use separation distance and truncation estimates.
Iterate with successive layers till ρ = r.o.c. is detected.

3. *Collection.* Translate each singular element by singularity analysis.

Yet to be implemented!!!

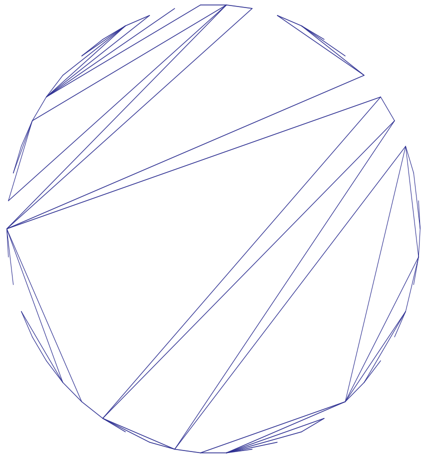
Algorithm ROCPAF (radius of conv., positive alg. fun.)
$Y \gg 0$ ($[z^n]Y \geq 0$); $P(z, Y) = 0$.

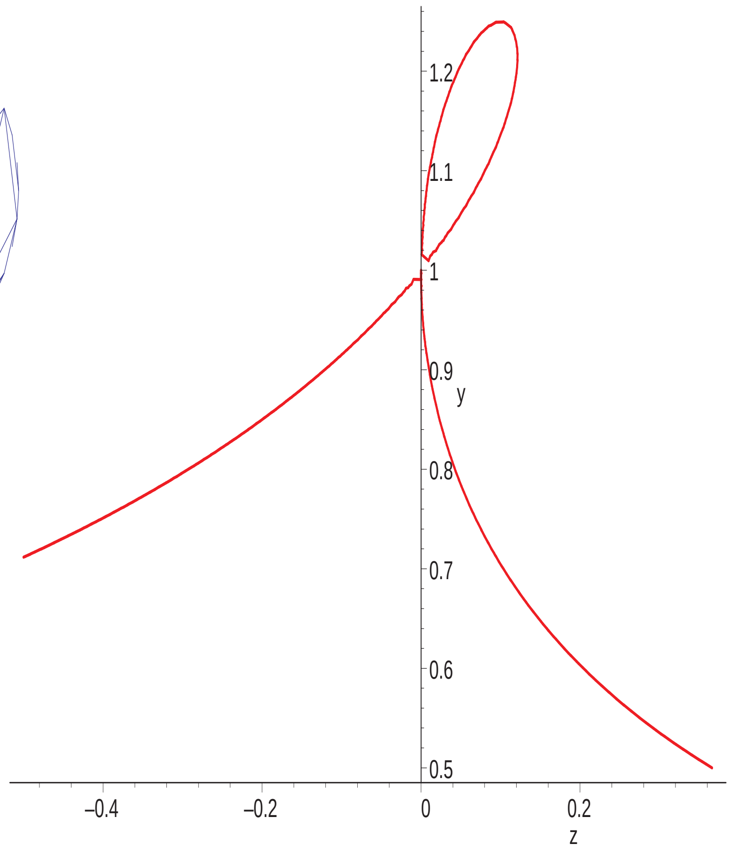
Prigsheim's Theorem: a positive dominant singularity.

Use SORTING of real branches based on local Puiseux expansions and a plane sweep.

Non-crossing forests, With **Maple/Gfun**

```
> # EXCEPTIONAL SET
> factor(resultant(P,diff(P,y),y));
             3         2      3
           -z  (4 - 8 z  + 5 z  - 32 z)

> fsolve(%,z,complex);
 -1.930283307, 0, 0, 0, .1215851069, 3.408698200

> P:=y^3+(z^2-z-3)*y^2+(z+3)*y-1;

          3     2           2
    P := y  + (z  - z - 3) y  + (z + 3) y - 1

> # PUISEUX EXPANSION AT 0 by GFUN (!)
> algeqtoseries(P,z,y,6);
            2      8/3        1/2      4/3
[1 + z + 2 z  + O(z   ), 1 - z    + O(z   ),

         1/2      4/3
    1 + z    + O(z   )]

> algeqtoseries(subs(z=z+sqrt(2),P),y,4): evalf(%);
                                                2
.2593842053 - .1136868103 z + .05214226099 z  - ...
```





Non-crossing graphs: (a) a random connected graph of size 50; (b) the real algebraic curve corresponding to non-crossing forests.

$$\xi \quad : \qquad \text{Root of} \quad 5z^3 - 8z^2 - 32z + 4 = 0$$

$$\beta = \frac{1}{37}\sqrt{228 - 981\xi - 5290\xi^2} \doteq 0.14931$$

$$F_n = \frac{\beta}{2\sqrt{\pi n^3}}\omega^n \left(1 + O(\frac{1}{n})\right), \qquad \omega = \frac{1}{\xi} \doteq 8.22469$$

The Drmota-Lalley-Woods Theorem

$$\{y_j = \Phi_j(z, y_1, \ldots, y_m)\}, \quad j = 1, \ldots, m.$$

Theorem [DLW]. Assume positive irreducible system.

All y_j have same dominant singularity ρ.
$\exists$ functions h_j analytic at the origin such that

$$y_j = h_j\left(\sqrt{1 - z/\rho}\right) \qquad (z \to \rho^-).$$

All other dominant sing. of the form $\rho\omega^j$, with ω root of unity—this, iff strongly periodic.

Asymptotics of the form (single sing.)

$$[z^n]y_j(z) \sim \rho^{-n}\left(\sum_{k\geq 1} d_k n^{-1-k/2}\right).$$

Proof (DLW)

Linearize by differentiating a certain number of times.
"Everybody" must have the same r.o.c by irreducibility.

Look at Jacobian of nonlinear transformation $\vec{\Phi}$;
Perron-Frobenius properties apply.

Look locally at what goes and exclude any behaviour other than square-root type.

Explains the ubiquitous $A^n n^{-3/2}$.

What else?

• Implement in a Computer Algebra Package [Chabaud]

• Find more structural theorems, extending DLW to reducible cases.

• Find "good subclasses"
— Trees, Walks, Maps are OK
— Others??

• Limit laws
— Gaussian-ness by perturbation and quasipowers: find "good" elegant conditions
— Classify possible nongaussian laws: cf Pemantle; cf stable laws [BaFlScSa01] etc.