Zeta Function Expansions of Classical Constants

Philippe Flajolet and Ilan Vardi

February 24, 1996

Many mathematical constants are expressed as slowly convergent sums of the form

$$C = \sum_{n \in A} f(\frac{1}{n}), \tag{1}$$

for some well-behaved analytic function f and some "reasonable" subset A of the integers. The convergence of such sums can be accelerated easily once values at the integers of the zeta function

$$\zeta_A(s) := \sum_{n \in A} \frac{1}{n^s} \tag{2}$$

are known: we have, formally at least,

$$C = \sum f_m \zeta_A(m) \quad \text{where} \quad f(z) = \sum_m f_m z^m \tag{3}$$

is the Taylor expansion of f at 0. This scheme is especially effective in the context of high-precision evaluation of mathematical constants as it often exhibits geometric convergence. It is also very easy to implement in current symbolic manipulation systems that have built in many mechanisms for the fast computation of zeta values.

In this note, we show the application of the rearrangement summarized by (1-3) in the case where A is either the whole set of integers, the prime numbers, or a congruence subset of the primes. This yields fast numerical schemes for the evaluation of many constants like: the number π, Euler's constant γ, Khinchin's constant K, the Hafner-Sarnak-McCurley constant σ, Hardy-Littlewood's twin prime constant H, or the Landau-Ramanujan constant λ. The reader is directed to Finch's beautiful pages [4] for background information on these and other classical constants.

§*1. The number π and Gregory's formula.* Many classical constants can be defined by a series of the form

$$S = \sum_{n=1}^{\infty} f(\frac{1}{n}), \tag{4}$$

where $f(z)$ is analytic at 0 and $f(z) = O(z^2)$ there. If $f(z)$ is analytic the closed unit disk, then rearranging the series gives

$$\begin{aligned} S &= \sum_{n=1}^{\infty}\sum_{m=2}^{\infty} f_m(\frac{1}{n})^m \\ &= \sum_{m=2}^{\infty}\sum_{n=1}^{\infty} f_m(\frac{1}{n})^m \\ &= \sum_{m=2}^{\infty} f_m \zeta(m) \end{aligned} \tag{5}$$

where

$$f(z) = \sum_{m=2}^{\infty} f_m z^m$$

is the Taylor expansion of f and $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the Riemann zeta function. While the original series (4) is slowly convergent, the transformed series (5) exhibits geometric convergence.

Similarly, if $f(z)$ is analytic in $|z| \leq 1/n_0$ for some positive integer n_0, then

$$S = \sum_{n=1}^{n_0-1} f(\frac{1}{n}) + \sum_{m=2}^{\infty} f_m \left[\zeta(m) - \frac{1}{1^m} - \cdots - \frac{1}{(n_0-1)^m}\right], \tag{6}$$

a formula that again converges geometrically: let $r > 1/n_0$ be such that $f(z)$ is analytic in $|z| \leq r$, then the "speed" of convergence is $O((rn_0)^{-m})$.

In addition, formula (6) may be used to accelerate the convergence of series like (4,5) by choosing an n_0 that is suitably large. This gives a whole range of formulæ that are intermediate between the original (4) and the transformed (5).

Vardi's book [9, p. 156] gives as an example Gregory's series for π,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty}\left(\frac{1}{4n-3} - \frac{1}{4n-1}\right),$$

whose rearrangement is found to be

$$\pi = \sum_{m=1}^{\infty} \frac{3^m - 1}{4^m}\zeta(m+1).$$

The same technique applies to Catalan's constant,

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{(4n-3)^2} - \frac{1}{(4n-1)^2} \right),$$

for which

$$G = \frac{1}{16} \sum_{m=1}^{\infty} (m+1) \frac{3^m - 1}{4^m} \zeta(m+2),$$

since

$$\frac{1}{(1-3z)^2} - \frac{1}{(1-z)^2} = \sum_{m=1}^{\infty} (m+1) \frac{3^m - 1}{4^m} z^m.$$

Given the formulæ (5,6), a sum like (4) is "easily" computable, once values of the zeta function at the integers have been tabulated, assuming that the Taylor expansion of f is itself easily computable. The table of zeta values can be built from the Euler-Maclaurin formula [3] while zetas of even argument are computable directly from Bernoulli numbers. In this way, Stieltjes [8, vol. II, p. 100] determined in 1887 the values $\zeta(2), \ldots, \zeta(70)$ to 30 digits of accuracy.

§*2. Euler's constant.* By definition, Euler's constant is

$$\gamma := \lim_{n \to \infty} \left(H_n - \log n \right).$$

The limit definition transforms into a sum,

$$\gamma = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} + \log \frac{n}{n+1} \right),$$

whose general term converges like $O(n^{-2})$. The series rearrangement now applies upon taking $n_0 = 2$,

$$\gamma = \frac{3}{2} - \log 2 - \sum_{m=2}^{\infty} (-1)^m \frac{m-1}{m} \left[\zeta(m) - 1 \right].$$

This is one of the many ways to find a geometrically convergent scheme for the computation of Euler's constant. An equivalent formula was already known to Euler and Stieltjes used precisely the formula corresponding to $n_0 = 3$ to check his computation of zeta values.

§*3. Khinchin's constant.* By taking logarithms, infinite products can also be computed. For instance, Vardi's book [9, p. 163] discusses the computation of Khinchin's constant

$$K = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right)^{\log n / \log 2}$$

along these lines. There, the general scheme applies with minor adjustments: because of the logarithms in the exponent, the rearranged series of $\log K$ involves the values $\zeta'(m)$. Other rearrangements, like

$$\log K = \frac{1}{\log 2} \sum_{m=1}^{\infty} \frac{h_{m-1}}{m} (\zeta(2m) - 1), \quad h_m = \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j}$$

(due to Shanks and Wrench, 1959) are known that do not require a special computation of the derivatives. However, Vardi's method has the advantage of complete generality as it applies to sums and products of the form

$$\sum_{n \geq 2} (\log n) f(\frac{1}{n}), \quad \prod_{n \geq 2} \left(1 + h(\frac{1}{n})\right)^{\log n},$$

of which Khinchin's constant is a particular instance.

§*4. Sums over primes and the Hafner-Sarnak-McCurley constant.* A sum of the form

$$T = \sum_{p \geq 2} f(\frac{1}{p}), \tag{7}$$

where by usual conventions an index p ranges over the prime numbers, can be rearranged into

$$T = \sum_{m=2}^{\infty} f_m \Pi(m), \tag{8}$$

provided f is analytic in $|z| \leq 1$. There,

$$\Pi(s) := \sum_{p \geq 2} \frac{1}{p^s}. \tag{9}$$

If $f(z)$ is analytic in $|z| \leq 1/n_0$, one has

$$T = \sum_{n=1}^{n_0 - 1} f(\frac{1}{n}) + \sum_{m=2}^{\infty} f_m \left[\Pi(m) - \frac{1}{1^m} - \cdots - \frac{1}{(n_0 - 1)^m}\right], \tag{10}$$

again with geometric convergence.

This technique makes it possible to evaluate T efficiently. In effect, $\Pi(s)$ is related to the zeta function, as results from the Eulerian product,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^{-s}}\right)^{-1},$$

by taking logarithms,

$$\begin{aligned} \log \zeta(s) &= \sum_{k\geq 1} \frac{1}{k} \sum_{p\geq 2} \frac{1}{p^k s} \\ &= \sum_{k\geq 1} \frac{1}{k} \Pi(ks). \end{aligned}$$

The last formula is a clear case of application of Moebius inversion and one finds

$$\Pi(s) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \zeta(ks). \tag{11}$$

In summary, T is efficiently computable from the formula (8) (or its variant (10)) with $\Pi(s)$ given by (11) which permits to build a table of values of $\Pi(m)$. Alternatively, one can compute T directly from the zeta values by combining the Moebius formula for $\Pi(m)$ and the expansion of f:

$$T = \sum_{\nu\geq 2} g_\nu \log \zeta(\nu), \qquad g_\nu = \sum_{k=1}^{\nu} \frac{\mu(k)}{k} f_{\nu/k}. \tag{12}$$

Hafner, Sarnak, and McCurley [5] have shown that the probability that two $m\times m$ and $n\times n$ matrices, m, n large, have relatively prime determinants is

$$\sigma = \prod_p \left(1 - \left[1 - \prod_{n=1}^{\infty}(1 - 1/p^n)\right]^2\right)$$

(it is well-known that this is $6/\pi^2$ when $m = n = 1$). An application of the method to $\log \sigma$ [9, p. 174] gives an equivalent form

$$\sigma = \prod_{m\geq 2} \zeta(m)^{-a_m}$$

and, upon taking 100 factors, we get

$$\sigma \approx 0.35323637185499598454\ldots,$$

The speed of convergence is determined by singularity analysis [9, p. 258-261] and is here roughly 0.57^n.

§*5. The Twin-primes constant.* Hardy and Littlewood [7, p. 371] developed in 1923 a heuristic model for the distribution of prime pairs according to which the number of prime pairs $p, p+2$ with $p \le x$ must be asymptotic to

$$2H\frac{x}{(\log x)^2}, \quad H = \prod_{p\ge 3}\left(1 - \frac{1}{(p-1)^2}\right).$$

The same constant also occurs in heuristic models of the Goldbach problem [4].

The twin prime constant is a direct case of application of the method of (12). It is amusing to note that the coefficients g_ν that appear there are related to finite fields. Let I_n be the number of monic irreducible polynomials over the coefficient field $GF(2)$ with degree n. We have [1, sec. 3.3]

$$I_n = \frac{1}{n}\sum_{d\,|\,n}\mu(d)2^{n/d}.$$

On the other hand,

$$\log(1 - \frac{1}{(p-1)^2}) = -\sum_{m=1}^{\infty}(2^m - 2)\frac{1}{mp^m}.$$

Thus,

$$H = \prod_{n=2}^{\infty}\left(\zeta(n)(1-2^{-n})\right)^{-I_n},$$

which converges like $(\frac{2}{3})^n$. Using a cutpoint n_0 like in (6,10) gives a whole collection of formulæ of which

$$H = \frac{3}{4}\frac{15}{16}\frac{35}{36}\prod_{n=2}^{\infty}\left(\zeta(n)(1-2^{-n})(1-3^{-n})(1-5^{-n})(1-7^{-n})\right)^{-I_n}$$

has rate of convergence $\approx (\frac{11}{2})^{-n}$, thus giving approximately 3/4 of a digit per iteration.

A closely related example is Mertens's constant [7, p. 351],

$$B_1 = \gamma + \sum_p\left(\log(1-p^{-1}) + \frac{1}{p}\right),$$

that intervenes in the formula

$$\sum_{p\le x}\frac{1}{p}=\log\log x+B_1+o(1).$$

By the same device, we have

$$e^{B_1}=e^{\gamma}\prod_{m=2}^{\infty}\zeta(m)^{\mu(m)/m},$$

since

$$e^{z}=\prod_{m=1}^{\infty}(1-z^m)^{-\mu(m)/m}.$$

§*6. The Landau Ramanujan constant.* This constant is defined by

$$\lambda=\left(\frac{1}{2}\prod_r\frac{1}{1-r^{-2}}\right)^{1/2},$$

in which $\prod_r$ indicates a product where r ranges over the prime numbers that are congruent to 3 modulo 4. It is known from work by Landau in 1908 rediscovered later by Ramanujan and mentioned in his first letter to Hardy [2, Ch. 23], [6] that the number of integers less than x that are sums of two squares is asymptotic to

$$\lambda\frac{x}{\sqrt{\log x}}.$$

Ramanujan gave the (correct) approximate value $\lambda \doteq 0.764$.

The general problem behind this example is to estimate sums over primes that satisfy congruence restrictions. Here, we let p range over all primes, q range over primes $\equiv 1 \bmod 4$ and r range over primes $\equiv 3 \bmod 4$. Euler's product formula is

$$\zeta(s)=(1-2^{-s})^{-1}\prod_q(1-q^{-s})^{-1}\prod_r(1-r^{-s})^{-1}. \tag{13}$$

The function $L(s)$ defined by

$$L(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

also admits an Eulerian decomposition

$$L(s) = \prod_q (1-q^{-s})^{-1} \prod_r (1+r^{-s})^{-1}. \tag{14}$$

Thus, comparison of (13) and (14) yields

$$(1-2^{-s})\frac{\zeta(s)}{L(s)} = \prod_r \frac{1+r^{-s}}{1-r^{-s}}. \tag{15}$$

Taking logarithms in (15), we get

$$\log \prod_r \frac{1+r^{-s}}{1-r^{-s}} = \ell(s) \qquad \text{where} \qquad \ell(s) = \log\left((1-2^{-s})\frac{\zeta(s)}{L(s)}\right). \tag{16}$$

Introduce the "base" function

$$R(s) = \sum_r \frac{1}{r^s}.$$

A Taylor expansion of the left hand side of (16) leads to the relation

$$R(s) + \frac{1}{3}R(3s) + \frac{1}{5}R(5s) + \cdots = \frac{1}{2}\ell(s), \tag{17}$$

which is easily inverted by a variant of Moebius inversion

$$R(s) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n}\ell(ns), \tag{18}$$

where

$$\widehat{\mu}(n) = \begin{cases} \mu(n) & \text{if } n \equiv 1 \pmod 2 \\ 0 & \text{if } n \equiv 0 \pmod 2). \end{cases}$$

Let $f(z)$ be any function analytic at the origin with radius of convergence larger than 1/3:

$$f(z) = \sum_{n=2}^{\infty} f_n z^n.$$

Then interchange of summations provides the identity

$$\sum_r f(\frac{1}{r}) = \sum_{n=2}^{\infty} f_n R(n), \tag{19}$$

which expresses a sum over primes $r \equiv 3 \pmod 4$ in terms of values of the base function $R(s)$ at the positive integers. The function $R(s)$ is computable from (16,18):

$$R(s) = \frac{1}{2} \sum \frac{\widehat{\mu}(n)}{n} \log \left((1 - 2^{-s}) \frac{\zeta(s)}{L(s)} \right).$$

The values of $L(s)$ themselves can be obtained either from the relation to the Hurwitz zeta function,

$$L(s) = \frac{1}{4^s} (\zeta(s, 1/4) - \zeta(s, 3/4)),$$

(this is computable directly by the Euler-Maclaurin summation formula) or by reduction to Riemann zeta values in accordance with the scheme employed for Gregory's series and for Catalan's constant.

In the case of the Landau-Ramanujan constant, applying the general algorithm leads to a wonderful formula,

$$\lambda = \frac{1}{\sqrt{2}} \prod_{n=1}^{\infty} \left[\left(1 - \frac{1}{2^{2^n}} \right) \zeta(2^n)/L(2^n) \right]^{1/2^{n+1}}, \tag{20}$$

as results from elementary Moebius function identities. An alternative direct proof runs as follows. Let

$$g(s) = (1 - 2^{-s})\, \zeta(s)/L(s), \qquad f(s) = \prod_r \frac{1}{1 - r^{-s}}$$

then it is seen that

$$g(s) = \frac{f(s)^2}{f(s)}$$

from which it follows that

$$\frac{f(s)^2}{f(2^{k+1}s)^{1/2^k}} = g(s)\, g(2s)^{1/2} g(4s)^{1/4} \cdots g(2^k s)^{1/2^k}$$

proving the identity along with an estimate of the rate of convergence.

Because of the lacunary character of the expression (20), the computation is extremely fast and it takes only $6 \cdot 10^8$ machine cycles (6 seconds of CPU time of 1996!) to get 200 digits of λ by (20) in pari/gp, for instance,

$$\lambda = 0.764223653589220662990698731250092328116790541393409514 72 \cdots.$$

References

[1] Berlekamp, E. R. *Algebraic Coding Theory.* Mc Graw-Hill, 1968. Revised edition, 1984.

[2] Berndt, B. C. *Ramanujan's Notebooks, Part IV.* Springer Verlag, 1994.

[3] Edwards, H. M. *Riemann's Zeta Function.* Academic Press, 1974.

[4] Finch, S. Favorite mathematical constants. Available under World Wide Web at `http: //www.mathsoft.com/asolve/constant/constant.html`, 1995.

[5] Hafner, J. L., Sarnak, P., and McCurley, K. Relatively prime values of polynomials. In *Contemporary Mathematics* (1993), M. Knopp and M. Sheingorn, Eds., vol. 143.

[6] Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*, third ed. Chelsea Publishing Company, New-York, 1978. Reprinted and Corrected from the First Edition, Cambridge, 1940.

[7] Hardy, G. H., and Wright, E. M. *An Introduction to the Theory of Numbers*, fifth ed. Oxford University Press, 1979.

[8] Stieltjes, T. J. *Œuvres Complètes.* Springer Verlag, 1993. Edited by Gerrit van Dijk.

[9] Vardi, I. *Computational Recreations in Mathematica.* Addison Wesley, 1991.