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SINGULARITY ANALYSIS OF GENERATING FUNCTIONS*

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Abstract. This work presents a class of methods by which one can translate, on a term-by-term basis, an asymptotic expansion of a function around a dominant singularity into a corresponding asymptotic expansion for the Taylor coefficients of the function. This approach is based on contour integration using Cauchy's formula and Hankel-like contours. It constitutes an alternative to either Darboux's method or Tauberian theorems that appears to be well suited to combinatorial enumerations, and a few applications in this area are outlined.

Key words. asymptotic analysis, generating functions, combinatorial enumeration

AMS(MOS) subject classifications. 05, 40, 68

1. Introduction. Several applications in analysis, especially combinatorics, necessitate determining the asymptotic order of growth of coefficients of a function that is analytic at the origin. It has been recognized for a long time that the function's dominant singularities (the ones of smallest modulus) contain a great deal of information on the coefficients. This paper describes a very general method based on earlier works of ours (Odlyzko [1982], Flajolet and Odlyzko [1982]) that applies to functions of "moderate" variation. We restrict our attention to functions with a unique dominant singularity. (Functions with a *finite* number of singularities on their circle of convergence can also be treated by a direct extension of our methods, using composite integration contours.) By normalization, we may always assume that the dominant singularity occurs at $z = 1$, and we consider functions that satisfy, for some arbitrary real number α,

$$f(z) \approx (1-z)^{\alpha} \qquad (z \to 1).$$

Let f_n be a sequence of numbers, with a generating function $f(z) = \sum_{n \geq 0} f_n z^n$ that is analytic at the origin. In nonelementary cases that we encounter in combinatorial analysis, the f_n are only accessible via $f(z)$, and $f(z)$ itself is either explicitly defined by a closed-form expression or implicitly specified as the solution to a functional equation. (See, for instance, Comtet [1974], Goulden and Jackson [1983], Stanley [1986].) The problem is thus to obtain estimates for f_n from whatever analytic information is available on $f(z)$.

For example, the generating function of 2-regular graphs (Comtet [1974]) is

$$f(z) = \frac{e^{-z/2 - z^2/4}}{\sqrt{1-z}},$$

and the expansion of $f(z)$ at $z = 1$,

$$f(z) = e^{-3/4}\left[\frac{1}{\sqrt{1-z}} + \sqrt{1-z} + \cdots\right], \tag{1.1a}$$

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has a matching expansion for coefficients f_n as $n \to \infty$,

$$f_n \sim e^{-3/4}\left[\binom{n-\frac{1}{2}}{n}+\binom{n-\frac{3}{2}}{n}+\cdots\right] \qquad (1.1\text{b})$$
$$\sim e^{-3/4}\left[\frac{1}{\sqrt{\pi n}}+\frac{c}{\sqrt{n^3}}+\cdots\right].$$

Darboux's method is one way of achieving the term-by-term transition from (1.1a) to (1.1b). (It is succinctly described, along with Tauberian methods, in § 5.) The *smoothness condition* ensuring its validity is that the expansion (1.1a) can be pushed until an error term, sufficiently differentiable on the circle $|z| = 1$, is obtained (Henrici [1977], Bender [1974], Comtet [1974], Olver [1974]). We present here a general method that ensures the validity of an expansion like (1.1b), using only *order-of-growth* information on the remainder terms in the asymptotic expansion of $f(z)$ in a suitable domain of the complex plane. For instance, under suitable analytic conditions, an expansion

$$f(z) \sim \frac{1}{\sqrt{1-z}}\left(\frac{c_0}{\sqrt{\log(1-z)^{-1}}}+\frac{c_1}{\log(1-z)^{-1}} + \frac{c_2}{\sqrt{\log^3(1-z)^{-1}}}+\cdots\right) \qquad (z \to 1) \qquad (1.2\text{a})$$

"transfers" to coefficients as

$$f_n \sim \frac{1}{\sqrt{\pi n}}\left(\frac{c_0}{\sqrt{\log n}}+\frac{c_1}{\log n}+\frac{c_2'}{\sqrt{\log^3 n}}+\cdots\right) \qquad (n \to \infty) \qquad (1.2\text{b})$$

(where c_2', c_3', $\cdots$ depend only on c_0, c_1, $\cdots$), but there is no way of achieving this by Darboux's method, since a remainder term introduced in (1.2a) cannot be differentiable at $z = 1$ unless the expansion is trivial.

More generally, we provide sufficient conditions for the validity of the implications

(T1) $\qquad f(z)=O(g(z)) \Rightarrow f_n=O(g_n)$

(T2) $\qquad f(z)=o(g(z)) \Rightarrow f_n=o(g_n)$

(T3) $\qquad f(z)\sim g(z) \Rightarrow f_n\sim g_n.$

The conditions are that $g(z)$ should belong to a well-defined asymptotic scale $\mathscr{S}$, and that the asymptotic form for f should be valid in a suitable domain of the complex plane, which usually requires *analytic continuation* of $f(z)$ outside its circle of convergence. The asymptotic scale $\mathscr{S}$ we consider here contains functions of z of the form

$$g(z)=A\left(\frac{1}{1-z}\right) \quad \text{with } A(u)\sim u^{\beta}(\log u)^{\gamma}(\log\log u)^{\delta} \quad \text{as } (u\to\infty)$$

whose coefficients will be later proved to satisfy

$$g_n \sim \frac{1}{\Gamma(\beta)}\frac{A(n)}{n}.$$

We observe from this last equation that larger singular functions at $z = 1$ have larger Taylor coefficients. Thus, again under suitable conditions on $f(z)$, we are justified in applying the transfer principle from the first equation to the second equation in the pair

$$\textbf{(T0)}\qquad \begin{aligned} &f(z)=h_0(z)+h_1(z)+\cdots+h_k(z)+O(g(z)) \\ &\qquad\qquad \text{with } h_0(z)\gg\cdots\gg h_k(z)\gg g(z) \qquad (z\to 1) \\ &f_n=h_{0,n}+h_{1,n}+\cdots+h_{k,n}+O(g_n) \\ &\qquad\qquad \text{with } h_{0,n}\gg\cdots\gg h_{k,n}\gg g_n \qquad (n\to\infty) \end{aligned}$$

when the $h_j(z)$ and $g(z)$ are in $\mathscr{S}$. Asymptotic expansions in a large class translate term-by-term in a direct way.

From now on, we shall call transfer theorems of types **(T1)**, **(T2)**, **(T3)**, and **(T0)** by the more suggestive names of O-transfers, o-transfers, $\sim$-transfers, and Σ-transfers (read as "big O," "little o," "sim" and "sigma" transfers!). The most basic transfers are O-transfers. A refinement of the proof of a O-transfer will usually lead to a o-transfer. As a direct consequence of o-transfers, we obtain $\sim$-transfers since

$$f(z)\sim g(z) \quad \text{is equivalent to } f(z)=g(z)+o(g(z)).$$

As indicated in the previous paragraph, Σ-transfers follow directly from O-transfers for expansions with an $O(\cdot)$ remainder term as in **(T0)** and there is an obvious analogue for o-transfers where we have an $o(\cdot)$ remainder term. A side issue to be considered is the determination of coefficients of standard singular functions (here, the scale $\mathscr{S}$). Fortunately, that task can itself be carried out by methods rather similar to the proofs of transfer theorems.

Several of our statements are inspired by Tauberian theorems, most notably the Hardy–Littlewood theorem (Hardy and Littlewood [1914]), though our analysis is quite different since it is based on contour integration. It is clearly less deep, but it has the advantage of great technical simplicity. Tauberian theorems impose no condition on the function, but they require a priori "side conditions" on the coefficients (positivity, monotonicity, etc.). (See Titchmarsh [1939], Hardy [1949], Postnikov [1980].) Our method imposes conditions on the function in the complex domain, but no a priori condition on the coefficients. That approach is therefore quite adequate for obtaining expansions of the Σ-type, for which Tauberian methods tend to be of little help (side conditions are hard to establish on error terms). Furthermore, in the context of combinatorial enumerations, large classes of generating functions are expected to be analytically continuable since they are obtained as combinations of analytic functionals applied to entire functions. In that context, our conditions are seldom a limitation.

The paper is organized as follows. In § 2, we start with a restricted scale $\mathscr{S}_0$ that contains only functions of the form $(1-z)^\alpha$, where α is *any* (positive or negative) real number. The O-transfer theorem (Theorem 1) that covers all values of α requires analytic continuation of $f(z)$ in an angular domain outside its circle of convergence and validity of the asymptotic expansions there. It is proved—as are all other results in this paper—by choosing a suitable contour (reminiscent of Hankel contours) in Cauchy's integral formula for coefficients of analytic functions, and "integral splitting." A modification of the proof gives us o-transfers (Corollary 1), from which we deduce $\sim$-transfers (Corollary 2) and a variety of Σ-transfers of which Corollary 3 is only a typical example. These results, though later generalized, are treated in some detail, as they serve to introduce basic techniques without unnecessary complications.

Our next objective, in § 3, is to extend theorems to the larger class $\mathcal{S}$ that has logarithms and iterated logarithms. We first establish (Theorem 2 and Corollary 4) general O- and o-transfer results for that class. More precise asymptotic estimates require us to characterize the growth of Taylor coefficients of functions that belong to the scale $\mathcal{S}$ (Theorem 3). This is done by using Hankel contours and modifications of the integral splitting techniques employed in our earlier proofs. Of the variety of conceivable $\sim$-transfers and Σ-transfers, we state only Corollaries 5 and 6, which correspond to functions with a descending expansion involving powers of logarithms or iterated logarithms.

Section 4 discusses various possible extensions. When $\alpha < -1$, so that $f(z) \approx (1-z)^\alpha$ is "large" at its singularity, the conditions on the function to be analyzed can be weakened somewhat (Theorem 4). Also, a superset of $\mathcal{S}$ that includes functions of "slow variation" towards infinity is shown to be amenable to transfer methods.

Section 5 is a brief discussion of the relation to Darboux's method and Tauberian theorems. We finally conclude, in § 6, by sketching the transfer part of a few applications to combinatorial enumerations and the analysis of algorithms.

Relation to other works. The Hankel contours are classical tools in the study of the gamma and zeta functions (Whittaker and Watson [1927]). They appear to be well suited to extract asymptotic information on coefficients of analytic functions. They have been used in combinatorial applications in Odlyzko [1982] and Flajolet and Odlyzko [1982], and in another context (enumeration of polynomials over finite fields) by Car [1982], [1984]. Some results similar to ours, but requiring different analytic conditions, have been derived by Wong and Wyman [1974].

In analytic number theory, Hankel contours are useful in the study of Dirichlet series with algebraic singularities by means of Perron's formula, a typical example being the study of the coefficient of $[n^{-s}]$ in $(\zeta(s))^{1/2}$ (Selberg [1954], Hardy [1940, p. 62]). They are also useful in the inversion of Laplace or Mellin transforms with algebraic or logarithmic singularities (Doetsch [1955, p. 158]). Techniques of §§ 2 and 3 also bear some resemblance to the Watson–Doetsch lemma for Laplace transforms (Henrici [1977, p. 389]).

2. A basic transfer theorem. We let $\mathcal{S}_0$ be the class of functions $g_\alpha(z) = K(1-z)^\alpha$ for α a real number and K a constant. The Taylor coefficients of any member of $\mathcal{S}_0$ are known both exactly[1]

(2.1)
$$[z^n](1-z)^\alpha = \binom{n-\alpha-1}{n} = \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)}$$

and asymptotically from Stirling's formula ($\alpha \notin \{0, 1, 2, \cdots\}$):

(2.2)
$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left[1+\frac{\alpha(\alpha+1)}{2n}+\frac{\alpha(\alpha+1)(\alpha+2)(3\alpha+1)}{24n^2}\right.$$
$$+\frac{\alpha^2(\alpha+1)^2(\alpha+2)(\alpha+3)}{48n^3}$$
$$\left.+\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(15\alpha^3+30\alpha^2+5\alpha-2)}{5760n^4}+\cdots\right].$$

[1] We let, as usual, $[z^n]f(z)$ denote the coefficient of z^n in the Taylor expansion of $f(z)$: If $f(z) = \sum_n f_n z^n$, then $[z^n]f(z) = f_n$.

Thus the binomial coefficients (2.1), as well as their main asymptotic equivalents in (2.2), form an asymptotic scale. There is in fact a general form of (2.2).

PROPOSITION 1. *The binomial coefficients expressing* $[z^n](1-z)^\alpha$ *have an asymptotic expansion as* $n \to \infty$,

$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(1+\sum_{k\geqq 1}\frac{e_k^{(\alpha)}}{n^k}\right), \qquad \alpha \notin \{0,1,2,\cdots\}, \tag{2.3}$$

where

$$e_k^{(\alpha)} = \sum_{l=k}^{2k}(-1)^l \lambda_{k,l}(\alpha+1)(\alpha+2)\cdots(\alpha+l) \tag{2.4}$$

$$\textit{with} \sum_{k,l\geqq 0}\lambda_{k,l}v^k t^l = e^t(1+vt)^{-1-1/v}.$$

Proposition 1, although it would probably follow by close inspection of Stirling's formula, is most easily proved by techniques introduced in § 3, so that we delay the proof until then. We also observe, incidentally, that in (2.1)–(2.3) α may be complex: If $\alpha = \sigma + it$, we have

$$[z^n](1-z)^\alpha \sim \frac{n^{-\sigma-1}}{\Gamma(-\sigma-it)}[\cos(t\log n) - i\sin(t\log n)].$$

In that case, the main term in (2.2), (2.3) is of order $n^{-\sigma-1}$ and it is multiplied by a periodic function of $\log n$.

We now propose to prove a transfer condition of the O-type. We give the proof in some detail for two reasons: first, the implied constant in the O's are "constructive" and tight, a fact of independent interest; second, it serves as a guiding pattern for later deriving a variety of transfer conditions. We let $\Delta \equiv \Delta(\phi, \eta)$ denote the closed domain

$$\Delta(\phi,\eta) = \{z/|z| \leqq 1+\eta, |\mathrm{Arg}(z-1)| \geqq \phi\}, \tag{2.5}$$

where we take $\eta > 0$ and $0 < \phi < (\pi/2)$. This domain has the form of an indented disk depicted on Fig. 1(a).

THEOREM 1. *Assume that, with the sole exception of the singularity* $z = 1$, $f(z)$ *is analytic in the domain* $\Delta = \Delta(\phi, \eta)$, *where* $\eta > 0$ *and* $0 < \phi < (\pi/2)$. *Assume further that as z tends to* 1 *in* Δ,

$$f(z) = O(|1-z|^\alpha), \tag{2.6a}$$

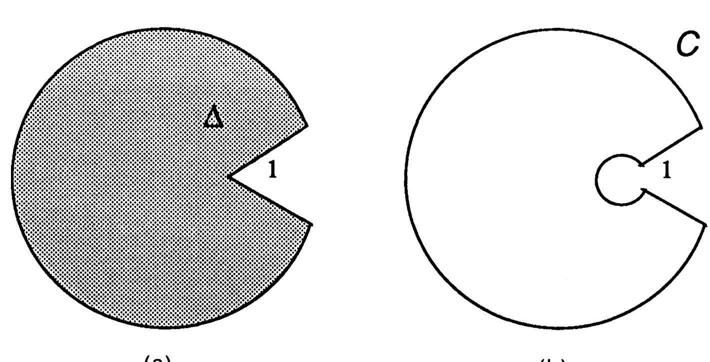

FIG. 1. (a) *The domain* $\Delta(\phi, \eta)$. (b) *The contour* $\mathscr{C}$ *used in the proof of Theorem* 1.

for some real number α. *Then the nth Taylor coefficient of* $f(z)$ *satisfies*

$$f_n=[z^n]f(z)=O(n^{-\alpha-1}). \tag{2.6b}$$

Proof. Since the modulus of $(1-z)^\alpha$ is bounded below by a constant >0 in any compact set in Δ that does not contain 1, and $f(z)$ is analytic in such a set, the *local* condition (2.6a) of the theorem is equivalent to assuming that for some constant $K>0$, we have in the *whole* of Δ with the possible exception of $z=1$,

$$|f(z)|<K|1-z|^\alpha. \tag{2.7}$$

In the derivation, we assume that $n \geqq 2|\alpha|+4$. This technical constraint is sufficient to ensure the validity of estimate (2.9) as well as the existence of the integrals appearing in (2.12), (2.13). We start from Cauchy's formula

$$f_n=\frac{1}{2i\pi}\int_{O^+} f(z)\frac{dz}{z^{n+1}}, \tag{2.8}$$

where O^+ is any positively oriented contour in Δ that encloses the origin, and we choose the (positively oriented) contour $\mathscr{C}=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ depicted in Fig. 1(b), with

$$\gamma_1=\left\{z/\ |z-1|=\frac{1}{n},\ |\mathrm{Arg}\,(z-1)|\geqq\phi\right\}$$

$$\gamma_2=\left\{z/\frac{1}{n}\leqq|z-1|,\ |z|\leqq 1+\eta,\ \mathrm{Arg}\,(z-1)=\phi\right\}$$

$$\gamma_3=\left\{z/\ |z-1|=1+\eta,\ |\mathrm{Arg}\,(z-1)|\geqq\phi\right\}$$

$$\gamma_4=\left\{z/\frac{1}{n}\leqq|z-1|,\ |z|\leqq 1+\eta,\ \mathrm{Arg}\,(z-1)=-\phi\right\}.$$

We proceed to evaluate the contributions to f_n due to each of the γ_j separately. So, we define

$$f_n^{(j)}=\frac{1}{2\pi}\int_{\gamma_j}|f(z)|\frac{|dz|}{|z|^{n+1}},$$

and we have $|f_n|\leqq f_n^{(1)}+f_n^{(2)}+f_n^{(3)}+f_n^{(4)}$.

1. *Smaller circle*. From (2.7), using trivial bounds on Cauchy's integral, we find as soon as $n\geqq 4$,

$$\begin{aligned} f_n^{(1)} &\leqq \left(\frac{1}{2\pi}\right)\cdot K\left(\frac{1}{n}\right)^{\alpha}\cdot\left(1-\frac{1}{n}\right)^{-n-1}\cdot\left(\frac{2\pi}{n}\right) \\ &\leqq 5\cdot(Kn^{-\alpha-1}). \end{aligned} \tag{2.9}$$

2. *Rectilinear part*. We next turn to the evaluation of $f_n^{(2)}$. By obvious symmetry considerations, the same bound will hold for $f_n^{(4)}$. We set $\omega=e^{i\phi}$ and perform the change of variable: $z=1+(\omega t/n)$. The definition of $f_n^{(2)}$ gives

$$\begin{aligned} f_n^{(2)} &\leqq \frac{1}{2\pi}\int_1^{En} K\left(\frac{t}{n}\right)^{\alpha}\left|1+\frac{\omega t}{n}\right|^{-n-1}\frac{dt}{n} \\ &\leqq (Kn^{-\alpha-1})\cdot\frac{1}{2\pi}\int_1^{\infty} t^{\alpha}\left|1+\frac{\omega t}{n}\right|^{-n-1}dt. \end{aligned} \tag{2.10}$$

Here E is defined so that γ_2 and γ_3 join: E is the positive root of $|1 + Ee^{i\phi}| = 1 + \eta$. We need to prove that the last integral in (2.10) is bounded above independently of n. First observe that

$$\left|1+\frac{\omega t}{n}\right| \geqq 1+\Re\left(\frac{\omega t}{n}\right)=1+\frac{t}{n}\cos\phi. \tag{2.11}$$

Thus, from (2.10) and (2.11), we find

$$f_n^{(2)} < \frac{J_n}{2\pi}(Kn^{-\alpha-1}) \quad \text{where } J_n=\int_1^\infty t^\alpha\left(1+\frac{t\cos\phi}{n}\right)^{-n} dt. \tag{2.12}$$

It is now easy to see that as $n \to \infty$,

$$J_n \to \int_1^\infty t^\alpha e^{-t\cos\phi}\, dt$$

and hence all the J_n are bounded above by some constant which depends only on α and ϕ. In fact, for positive λ, function $(1 + \lambda/n)^{-n}$ is a monotone decreasing function of n. In summary, we have proved that

$$f_n^{(2)} < \frac{J(\alpha,\phi)}{2\pi}(Kn^{-\alpha-1}) \quad \text{where } J(\alpha,\phi)=\int_1^\infty t^\alpha\left(1+\frac{t\cos\phi}{\nu}\right)^{-\nu} dt \tag{2.13}$$

and $\nu = 2|\alpha| + 4$.

The fact that ϕ is strictly less than $\pi/2$, whence $\cos\phi > 0$, is obviously crucial to this part of the analysis.

3. *Larger circle*. The majorization of $f_n^{(3)}$ gives an exponentially decreasing term:

$$\begin{aligned} f_n^{(3)} &< \frac{1}{2\pi}\cdot K\eta^\alpha\cdot(1+\eta)^{-n-1}\cdot(2\pi(1+\eta)) \\ &< K\frac{\eta^\alpha}{(1+\eta)^n}. \end{aligned} \tag{2.14}$$

4. Collecting the results of (2.9), (2.13), and (2.14), we have proved, for all $n \geqq 2|\alpha| + 4$:

$$f_n < (Kn^{-\alpha-1})\left[5+\frac{J(\alpha,\phi)}{\pi}+\frac{\eta^\alpha}{(1+\eta)^n}n^{\alpha+1}\right]. \tag{2.15}$$

There is an effectively computable constant n_1 (only depending on α and η), such that, for all $n \geqq n_1$,

$$\frac{\eta^\alpha}{(1+\eta)^n}n^{\alpha+1} < 1. \tag{2.16}$$

Thus, from (2.15), (2.16), we obtain our main bound,

$$f_n < (Kn^{-\alpha-1})\cdot\left[6+\frac{J(\alpha,\phi)}{\pi}\right] \quad \text{for all } n \geqq n_0, \tag{2.17}$$

where $n_0 = \max(n_1, 2|\alpha| + 4)$. Equation (2.17) is a stronger form of the statement of the theorem. □

The idea of a contour that "goes away" from the singularity at an angle has the essential feature of introducing, in Cauchy's integral, a "kernel" (z^{-n} or $(1 + t/n)^{-n}$) that decreases very fast along the contour, so that it captures the dominant contribution from an immediate vicinity of the singularity.

The proof techniques of Theorem 1, slightly modified, will give us a transfer result of o-type, from which we immediately deduce $\sim$- and Σ-transfers. The results that follow are not strictly speaking corollaries of Theorem 1, but rather of the line of proof taken there.

COROLLARY 1. *Assume that $f(z)$ is analytic in $\Delta\setminus\{1\}$, and that as $z \to 1$ in Δ,*

$$f(z) = o((1-z)^{\alpha}).$$

Then, as $n \to \infty$,

$$f_n = o(n^{-\alpha-1}).$$

Proof. The proof is an "ε-δ exercise." We use the same contour $\mathscr{C}$ as in Theorem 1. Observe again that there exists a $K > 0$ such that in the whole of Δ, $|f(z)| < K|1 - z|^{\alpha}$. By the o hypothesis on $f(z)$, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for z in Δ,

$$|z-1| < \delta \quad \Rightarrow \quad |f(z)| < \varepsilon|1-z|^{\alpha}.$$

We need to prove, with some fixed constant $K' > 0$, that for any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that

$$|f_n| < \varepsilon K' n^{-\alpha-1} \quad \text{whenever } n \geqq n_0(\varepsilon).$$

In the following derivations, the constants also depend on α, ϕ, and η, but we shall only indicate the dependence on ε. We choose a fixed (but arbitrary) ε, with its associated $\delta = \delta(\varepsilon)$.

1. *Smaller circle*. For the part γ_1 of the contour, we first choose $n_1 = n_1(\varepsilon)$ such that $1/n_1 \leqq \delta(\varepsilon)$. This choice ensures that part γ_1 of the contour is inside a domain where $|f(z)| < \varepsilon|1 - z|^{\alpha}$, and thus for $n > n_1$, as in (2.9),

$$f_n^{(1)} < 5(\varepsilon n^{-\alpha-1}). \tag{2.18}$$

2. *Rectilinear part*. Following the lines of (2.10), (2.11), with $z = 1 + \omega t/n$, we have

$$f_n^{(2)} < \frac{1}{2\pi}\int_1^{En} \left|f\left(1+\frac{\omega t}{n}\right)\right|\left(1+\frac{t\cos\phi}{n}\right)^{-n}\frac{dt}{n}. \tag{2.19}$$

We now decompose the integral in (2.19), and set $f_n^{(2)} = f_n^{(21)} + f_n^{(22)}$, where

$$f_n^{(21)} = \frac{1}{2\pi}\int_1^{\log^2 n} \quad \text{and} \quad f_n^{(22)} = \frac{1}{2\pi}\int_{\log^2 n}^{En}.$$

We can choose $n_{21} = n_{21}(\varepsilon)$ such that for all $n \geqq n_{21}$, we have $\log^2 n/n < \delta(\varepsilon)$, and then $1 + \omega t/n$ is in the "epsilon region" of f. Thus

$$\begin{aligned} f_n^{(21)} &< \frac{1}{2\pi}\int_1^{\log^2 n} \varepsilon\left(\frac{t}{n}\right)^{\alpha}\left(1+\frac{t\cos\phi}{n}\right)^{-n}\frac{dt}{n} \\ &< (\varepsilon n^{-\alpha-1})\cdot\frac{1}{2\pi}\int_1^{\infty} t^{\alpha}\left(1+\frac{t\cos\phi}{n}\right)^{-n} dt. \end{aligned} \tag{2.20}$$

From the argument used for (2.13), we find

$$f_n^{(21)} < (\varepsilon n^{-\alpha-1}) \frac{J(\alpha,\phi)}{2\pi}. \tag{2.21}$$

By similar devices, we get for $f_n^{(22)}$ the bound

$$f_n^{(22)} < (K n^{-\alpha-1}) \frac{1}{2\pi} \int_{\log^2 n}^{En} t^{\alpha} \left(1 + \frac{t\cos\phi}{n}\right)^{-n} dt. \tag{2.22}$$

The integrand in (2.22) is already exponentially small at $t = \log^2 n$. Without loss of generality, we may assume (by taking η small enough) that $E < 1/(4 \cos \phi)$. This guarantees $u = t \cos \phi / n < \frac{1}{4}$, so that, in the given range of values of t, $\log(1 + u) > u/2$ and

$$\left(1 + \frac{t\cos\phi}{n}\right)^{-n} < \exp\left(-\frac{\cos\phi}{2}\log^2 n\right).$$

Thus the integral in (2.22) is, say, $< n^{-2}$ for $n \geqq n_{22}$, and so

$$f_n^{(22)} < \frac{1}{n^2}(K n^{-\alpha-1}). \tag{2.23}$$

3. *Larger circle*. Finally, for $f_n^{(3)}$, we just use the previously established bound (2.14), which we repeat here,

$$f_n^{(3)} < K \frac{\eta^{\alpha}}{(1+\eta)^n}. \tag{2.24}$$

4. Collecting the results of (2.18), (2.21), (2.23), and (2.24), we find that, for all $n \geqq n_4(\varepsilon)$, where $n_4 = \max(n_1, n_{21}, n_{22})$,

$$n^{\alpha+1} f_n < \varepsilon \left[5 + \frac{J(\alpha,\phi)}{\pi}\right] + K\left[\frac{2}{n^2} + \frac{\eta^{\alpha}}{(1+\eta)^n} n^{\alpha+1}\right]. \tag{2.25}$$

We can obviously choose an $n_5 = n_5(\varepsilon)$ such that for all $n \geqq n_5$,

$$K\left(\frac{2}{n^2} + \frac{\eta^{\alpha}}{(1+\eta)^n} n^{\alpha+1}\right) < \varepsilon.$$

Then for $n \geqq n_0$ where $n_0 = n_0(\varepsilon) = \max(n_4, n_5)$, we obtain

$$n^{\alpha+1} f_n < \varepsilon \cdot \left[6 + \frac{J(\alpha,\phi)}{\pi}\right], \tag{2.26}$$

and (2.26) yields our corollary. □

We can now conclude this section by stating a $\sim$-transfer and a Σ-transfer.

COROLLARY 2. *Assume that $f(z)$ is analytic in $\Delta \setminus \{1\}$, and that as $z \to 1$ in Δ,*

$$f(z) \sim K(1-z)^{\alpha}.$$

Then, as $n \to \infty$: (i) *If $\alpha \notin \{0, 1, 2, \cdots\}$,*

$$f_n \sim \frac{K}{\Gamma(-\alpha)} n^{-\alpha-1};$$

(ii) *If α is a nonnegative integer, then*

$$f_n = o(n^{-\alpha-1}).$$

Proof. It suffices to apply Corollary 1 to the expansion

$$f(z)=K(1-z)^{\alpha}+o((1-z)^{\alpha}). \qquad \square$$

COROLLARY 3. *Assume that $f(z)$ is analytic in $\Delta\setminus\{1\}$, and that as $z\to 1$ in Δ,*

$$f(z)=\sum_{j=0}^{m} c_j(1-z)^{\alpha_j}+O((1-z)^{A}) \tag{2.27a}$$

where $\Re(\alpha_0)\leqq\Re(\alpha_1)\leqq\cdots\leqq\Re(\alpha_m)<A$. Then as $n\to\infty$,

$$f_n=\sum_{j=0}^{m} c_j\binom{n-\alpha_j-1}{n}+O(n^{-A-1}). \tag{2.27b}$$

Proof. The proof is a direct consequence of Theorem 1. $\square$

By Proposition 1, expansion (2.27b) can in turn be converted into another asymptotic expansion

$$f_n=\sum_{j=0}^{m'} c'_j n^{-\alpha'_j-1}+O(n^{-A-1}), \tag{2.27c}$$

where the α'_j belong to $\{\alpha_0,\cdots,\alpha_m\}+\{0,1,2,3,\cdots\}$ and satisfy $\Re(\alpha'_0)\leqq\cdots\Re(\alpha_{m'})<A$. Obviously, the method applies to a large class of asymptotic expansions by "subtracting singularities." For instance, from

$$f(z)=\log\frac{1}{1-z}+c_0+c_1(1-z)^{1/4}+c_2(1-z)^{1/2}+o((1-z)^{1/2}), \tag{2.28a}$$

we observe that $f(z)-\log(1-z)^{-1}=c_0+c_1(1-z)^{1/4}+\cdots$, and derive

$$f_n=\frac{1}{n}\left[1+\frac{c_1}{\Gamma(-\frac{1}{4})}n^{-1/4}+\frac{c_2}{\Gamma(-\frac{1}{2})}n^{-1/2}+o(n^{-1/2})\right]. \tag{2.28b}$$

3. A general asymptotic scale. The approach that gave us transfer results for functions of type $(1-z)^{\alpha}$ (the asymptotic scale $\mathscr{S}_0$) can be easily extended to cover a larger class $\mathscr{S}$ of singular functions, which we take to be of the form

$$g(z)=K(1-z)^{\alpha}\cdot L\left(\frac{1}{1-z}\right)\quad\text{where } L(u)=(\log u)^{\gamma}(\log\log u)^{\delta}. \tag{3.1}$$

Essentially, our previous results hold true provided we add an extra factor of $L(n)$ in asymptotic formulæ for coefficients. It should be clear from the derivations that our results depend on the fact that functions $L(u)$ "vary slowly" towards infinity, so that they behave almost like constants in our proofs: The key property is that $L(u)$ should satisfy

$$\frac{L(\lambda e^{i\theta}u)}{L(u)}\to 1,\qquad (u\to+\infty),$$

in a suitably uniform way for any fixed $\lambda>0$ and $|\theta|\leqq\pi-\phi$ (see § 4 for a more complete discussion).

THEOREM 2. *Assume that, with the sole exception of the singularity $z=1$, $f(z)$ is analytic in the domain $\Delta=\Delta(\phi,\eta)$, where $\eta>0$ and $0<\phi<(\pi/2)$. Assume further that as z tends to 1 in Δ,*

$$f(z)=O\left((1-z)^{\alpha}L\left(\frac{1}{1-z}\right)\right)\quad\textit{where } L(u)=(\log u)^{\gamma}(\log\log u)^{\delta}, \tag{3.2a}$$

for some real numbers α, γ, δ. Then the nth Taylor coefficient of $f(z)$ satisfies

$$f_n = [z^n]f(z) = O(n^{-\alpha-1}L(n)). \tag{3.2b}$$

Proof. Logarithms are taken with their principal determination. Without loss of generality, we may assume that η is small enough. Evaluation of the various contributions to Cauchy's integral proceeds very much like the evaluation in the proof of Corollary 1.

1. *Smaller circle.* Using trivial bounds, we find with $z = 1 - e^{i\theta}/n$, and θ varying in $[-(\pi-\phi), \pi-\phi]$,

$$f_n^{(1)} = O(n^{-\alpha-1}M_1(n)) \quad \text{where } M_1(n) = \sup_\theta |L(ne^{-i\theta})|.$$

It is easy to see that $L(u)$ does not vary much along any arc of a large circle centered at the origin, so that $M_1(n) \sim L(n)$ as $n \to \infty$, and

$$f_n^{(1)} = O(n^{-\alpha-1}L(n)). \tag{3.3}$$

2. *Rectilinear contour.* We set $z = 1 + \omega t/n$, and use the same splitting as in Corollary 2: $f_n^{(2)} = f_n^{(21)} + f_n^{(22)}$. First, we have

$$f_n^{(21)} = O\left(n^{-\alpha-1}M_2(n)\int_1^{\log^2 n} t^\alpha \left(1+\frac{t\cos\phi}{n}\right)^{-n} dt\right) \tag{3.4}$$
$$\text{where } M_2(n) = \sup_{t\in[1,\log^2 n]} L\left(-\frac{n}{\omega t}\right).$$

Again, the "slow variation" of $L(u)$ towards infinity entails that $M_2(n) \sim L(n)$. The integral in (3.4) is $O(1)$, hence

$$f_n^{(21)} = O(n^{-\alpha-1}L(n)). \tag{3.5}$$

We now turn to $f_n^{(22)}$, for which we have

$$f_n^{(22)} < \frac{Kn^{-\alpha-1}}{2\pi}\int_{\log^2 n}^{En} t^\alpha \left|L\left(-\frac{n}{\omega t}\right)\right| \left(1+\frac{t\cos\phi}{n}\right)^{-n} dt. \tag{3.6}$$

When z is on γ_2, quantity $u(t) = -n/\omega t$ goes from an area (for $t = En$) where it is $O(1)$ to a neighbourhood of infinity as $e^{-i(\pi-\phi)}\infty$ (for $t = \log^2 n$). Over $u(\gamma_2)$, function $L(u)$ is upperbounded by a linear function of $|u|$. (Actually, it cannot grow faster than $|u|^\varepsilon$). We thus have $|L(u)| < K_1|u| + K_2$, and

$$f_n^{(22)} < \frac{n^{-\alpha-1}}{2\pi}\int_{\log^2 n}^{En} t^\alpha \left(K_1\left|\frac{n}{t}\right| + K_2\right)\left(1+\frac{t\cos\phi}{n}\right)^{-n} dt.$$

If we use the crude bound $K_1|n/t| + K_2 = O(n)$, we find,

$$f_n^{(22)} = \frac{n^{-\alpha-1}}{2\pi}\cdot O(n)\cdot\int_{\log^2 n}^{En} t^\alpha\left(1+\frac{t\cos\phi}{n}\right)^{-n} dt. \tag{3.7}$$

By an argument already encountered in (2.22), (2.23), the integral in (3.7) is $O(1/n^2)$, so that finally

$$f_n^{(22)} = O(n^{-\alpha-2}). \tag{3.8}$$

3. *Larger circle*. Again, we only need to use

$$f_n^{(3)} = O((1+\eta)^{-n}). \tag{3.9}$$

4. Collecting the bounds from (3.3), (3.5), (3.8), and (3.9), we find

$$\begin{aligned} f_n &= O(L(n)n^{-\alpha-1}) + O(n^{-\alpha-2}) + O((1+\eta)^{-n}) \\ &= O(L(n)n^{-\alpha-1}), \end{aligned} \tag{3.10}$$

since $1/n$ is always $o(L(n))$ as $n \to \infty$. □

A slight modification of the proof of Theorem 2 (see also Corollary 2) yields the following corollary.

COROLLARY 4. *Assume that $f(z)$ is analytic in $\Delta(\phi, \eta) \setminus \{1\}$ and that as $z \to 1$ in Δ*

$$f(z) = o\left((1-z)^\alpha L\left(\frac{1}{1-z}\right)\right) \quad \textit{where } L(u) = (\log u)^\gamma (\log\log u)^\delta, \tag{3.11a}$$

Then the nth Taylor coefficient of $f(z)$ satisfies

$$f_n = [z^n] f(z) = o(n^{-\alpha-1} L(n)). \tag{3.11b}$$

In order to proceed further, we need to find detailed asymptotic expansions for coefficients of a set of functions of the form (3.1), thereby generalizing the classical expansion of $(1-z)^\alpha$ that was stated in Proposition 1. There is only a minor technical difficulty, namely that the functions in (3.1) are not in general analytic at the origin, so that we operate with slightly modified functions. It will be recognized that this modification is of no consequence for asymptotic expansions of coefficients. (See the remarks following Corollary 5).

Our proof technique is based on the use of contours of Hankel type for the Gamma function.

THEOREM 3A. *Let α and γ be real or complex numbers, $\alpha, \gamma \notin \{0, 1, 2, \cdots\}$. Define the function $f(z)$ by*

$$f(z) = (1-z)^\alpha \left(\frac{1}{z} \log \frac{1}{1-z}\right)^\gamma.$$

Then, the Taylor coefficients of $f(z)$ satisfy

$$f_n = [z^n] f(z) \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^\gamma \left(1 + \sum_{k \geq 1} \frac{e_k^{(\alpha,\gamma)}}{\log^k n}\right),$$

with

$$e_k^{(\alpha,\gamma)} = (-1)^k \binom{\gamma}{k} \Gamma(-\alpha) \frac{d^k}{ds^k} \left(\frac{1}{\Gamma(-s)}\right)\bigg|_{s=\alpha}.$$

Proof. Observe that $f(z)$ is analytic in the plane slit along $[1, +\infty]$. We evaluate the Cauchy integral giving coefficients f_n along a contour (see Fig. 2(a)) $\mathscr{C} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where γ_3 is an arc of the circle with radius 2, the rest of the contour being an

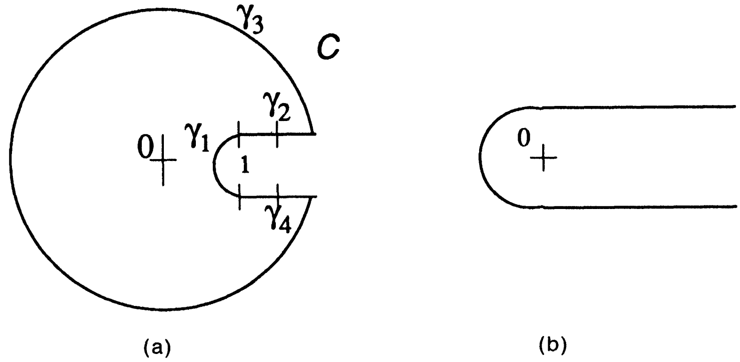

FIG. 2. *Various Hankel contours*: (a) *Contour* $\mathscr{C}$ *for the proof of Theorem* 3. (b) *Contour* H.

open loop around $[1, 2]$ at distance $1/n$. In symbols,

$$\gamma_1 = \left\{ z = 1 - \frac{t}{n} \Big/ t = e^{i\theta}, \theta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2} \right] \right\}$$

$$\gamma_2 = \left\{ z = 1 + \frac{t+i}{n} \Big/ t \in [0, n] \right\}$$

$$\gamma_3 = \left\{ z \Big/ |z| = \left(4 + \frac{1}{n^2} \right)^{1/2}, \Re(z) \leqq 2 \right\}$$

$$\gamma_4 = \left\{ z = 1 + \frac{t-i}{n} \Big/ t \in [0, n] \right\}.$$

We immediately dispose of the contribution to Cauchy's integral due to γ_3. It satisfies

$$f_n^{(3)} = O(2^{-n}), \tag{3.12}$$

and is thus exponentially small. Let $f_n^{(124)}$ denote the contribution from the rest of the contour, $h_1 = \gamma_1 \cup \gamma_2 \cup \gamma_4$. We perform the change of variable $z = 1 + t/n$, and let H_1 be the contour on which t varies: H_1 is an open loop at distance 1 from the segment $[0, n]$ of the positive real axis. We have

$$n^{\alpha+1} f_n^{(124)} = \frac{1}{2i\pi} \int_{H_1} (-t)^{\alpha} \left(\log \left(-\frac{n}{t} \right) \right)^{\gamma} \left(1 + \frac{t}{n} \right)^{-n-1-\gamma} dt. \tag{3.13}$$

Most of the contribution is expected to come from the area where $t \ll n$ because of the fast decreasing $(1 + t/n)^{-n}$ factor. Let H_1' be the part of the contour H_1 such that $|t| < \log^2 n$. Along $H_1 \backslash H_1'$, the integrand contains an exponentially small factor of the form $e^{-c \log^2 n}$, and is thus negligible. Along H_1', by devices that should now be familiar, we may use in (3.13) the approximation $(1 + (t/n))^{-n} \approx e^{-t}$. In this way, we get

$$n^{\alpha+1} f_n^{(124)} = \frac{1}{2i\pi} \int_{H_1'} (-t)^{\alpha} \left(\log \left(-\frac{n}{t} \right) \right)^{\gamma} e^{-t}\, dt + O\left(\frac{\log^{\gamma} n}{n} \right). \tag{3.14}$$

Still along H'_1, we have $\log t \ll \log n$, hence we can expand the logarithmic part of the integrand:

$$\begin{aligned}(3.15)\qquad \left(\log\left(-\frac{n}{t}\right)\right)^{\gamma} &= \log^{\gamma} n\left(1-\frac{\log(-t)}{\log n}\right)^{\gamma}\\ &= \log^{\gamma} n\left(\sum_{k=0}^{m-1}\binom{\gamma}{k}(-1)^k\left(\frac{\log(-t)}{\log n}\right)^k + O\left(\left(\frac{\log(-t)}{\log n}\right)^m\right)\right).\end{aligned}$$

We substitute expansion (3.15) inside the integral of (3.14), and get

$$\begin{aligned}(3.16)\qquad &\frac{n^{\alpha+1}f_n^{(124)}}{\log^{\gamma} n} = \sum_{k=0}^{m-1}\binom{\gamma}{k}(-1)^k\frac{I_k}{\log^k n} + O\left(\frac{1}{\log^m n}\right) \quad \text{where}\\ &I_k = \frac{1}{2i\pi}\int_{H'_1}(-t)^{\alpha}(\log(-t))^k e^{-t}\,dt.\end{aligned}$$

We can now extend the rectilinear parts of contour H'_1 towards $+\infty$ (see Fig. 2(b)). This gives us a new contour H, and the process introduces only exponentially small terms in the integral. In this way, we find from (3.16), which is valid for any $m \geqq 1$,

(3.17)

$$\frac{n^{\alpha+1}f_n^{(124)}}{\log^{\gamma} n} \sim \sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\frac{G_k}{\log^k n} \quad \text{where } G_k = \frac{1}{2i\pi}\int_H(-t)^{\alpha}(\log(-t))^k e^{-t}\,dt.$$

From the bound (3.12) for $f_n^{(3)}$, we see that the same expansion (3.17) holds for f_n. All that remains is to compute the G_k. But, by a familiar integral (Whittaker and Watson [1927, p. 245]), we have

$$G_0 \equiv \frac{1}{2i\pi}\int_H(-t)^{\alpha}e^{-t}\,dt = \frac{1}{\Gamma(-\alpha)},$$

and obviously G_k is the kth derivative of G_0 with respect to α. Our proof is now complete. □

THEOREM 3B. *Let α, γ, and δ be complex numbers not in $\{0, 1, 2, \cdots\}$. Define the function $f(z)$*

$$f(z) = (1-z)^{\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\gamma}\left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^{\delta}.$$

Then, the Taylor coefficients of $f(z)$ satisfy

$$f_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^{\gamma}(\log\log n)^{\delta}\left(1+\sum_{k\geqq 1}\frac{e_k(\log\log n)}{(\log n\log\log n)^k}\right),$$

where $e_k(x) = e_k^{(\alpha,\gamma,\delta)}(x)$ is a polynomial of degree k,

$$e_k(x) = \Gamma(-\alpha)\frac{d^k}{ds^k}\frac{1}{\Gamma(-s)}\bigg|_{s=\alpha}\cdot E_k(x) \quad \textit{with}$$

$$\sum_{k=0}^{\infty}E_k(x)u^k = (1-xu)^{\gamma}\left(1-\frac{1}{x}\log(1-xu)\right)^{\delta}.$$

Sketch of proof. The proof starts with the analogue of (3.14),

(3.18)

$$n^{\alpha+1} f_n = \frac{1}{2i\pi} \int_{H_1} (-t)^\alpha \left(\log \left(-\frac{n}{t} \right) \right)^\gamma \left(\log\log \left(-\frac{n}{t} \right) \right)^\delta e^{-t}\, dt + O\left(\frac{(\log n)^\gamma (\log\log n)^\delta}{n} \right)$$

and uses (3.15) together with

$$\left(\log\log \left(-\frac{n}{t} \right) \right)^\delta = (\log\log n)^\delta \left(1 + \frac{1}{\log\log n} \log \left(1 - \frac{\log(-t)}{\log n} \right) \right)^\delta . \tag{3.19}$$

The right-hand side of (3.19) can then be expanded in descending powers of $\log n$ and $\log\log n$, etc. $\square$

The same line of proof now enables us to prove the basic asymptotic estimate for the binomial coefficients already announced in Proposition 1.

Proof of Proposition 1. Using again a variant of (3.13), (3.14), we find that

$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \cdot \frac{1}{2i\pi} \int_{H_1} (-t)^\alpha \left(1 + \frac{t}{n} \right)^{-n-1} dt \tag{3.20}$$

and it is the expansion

$$\left(1 + \frac{t}{n} \right)^{-n-1} = e^{-t} \left(1 + \frac{t^2 - 2t}{2n} + \frac{6t^4 - 40t^3 + 48}{48n^2} + \cdots \right) \tag{3.21}$$

in descending powers of $1/n$ that provides an explicit form of the $e_k^{(\alpha)}$. $\square$

There are obvious $\sim$- and Σ-transfer results that follow from these equations. We only cite two simple analogues of Corollary 3.

COROLLARY 5. *Assume that* $f(z)$ *is analytic in* $\Delta \backslash \{1\}$, *and that as* $z \to 1$ *in* Δ,

$$f(z) = (1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\gamma \left[\sum_{j=0}^{m-1} c_j \left(\log \frac{1}{1-z} \right)^{-j} + O\left(\left(\log \frac{1}{1-z} \right)^{-m} \right) \right] \tag{3.22a}$$

for some $\alpha, \gamma \notin \{0, 1, 2, \cdots\}$. *Then as* $n \to \infty$,

$$f_n = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \log^\gamma n \left[\sum_{j=0}^{m-1} c'_j \log^{-j} n + O(\log^{-m} n) \right]. \tag{3.22b}$$

Proof. We only need to check that in expansion (3.22a), replacing

$$\left(\log \frac{1}{1-z} \right)^\gamma \quad \text{by} \quad \left(\frac{1}{z} \log \frac{1}{1-z} \right)^\gamma$$

introduces error terms that are of order $(1-z)^{\alpha-1}(\log(1-z))^\gamma$, using the expansion of $1/z$ at $z = 1$. We then conclude by an application of Theorem 3A and Theorem 2. The c'_j are computable from the c_j by the expansion of Theorem 3A. $\square$

COROLLARY 6. *Assume that* $f(z)$ *is analytic in* $\Delta \backslash \{1\}$, *and that as* $z \to 1$ *in* Δ,

(3.23a)

$$f(z) = (1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\gamma \left(\log\log \frac{1}{1-z} \right)^\delta \left[\sum_{j=0}^{m-1} c_j \left(\log\log \frac{1}{1-z} \right)^{-j} + O\left(\left(\log\log \frac{1}{1-z} \right)^{-m} \right) \right]$$

for some α, $\gamma \notin \{0, 1, 2, \cdots\}$. *Then as* $n \to \infty$,

$$(3.23\text{b}) \qquad f_n = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^{\gamma} (\log \log n)^{\delta} \left[\sum_{j=0}^{m-1} c_j (\log \log n)^{-j} + O((\log \log n)^{-m}) \right].$$

(The coefficients c_j are the same in both expansions.)

A few historical remarks on the ancestry of Theorem 3 and particular cases not yet explicitly covered are due here. We assumed in the statement of Theorem 3A that neither α nor γ are integers. This leaves three cases to be discussed that can also be treated by our methods.

1. The case where γ is an integer and α not an integer was studied by Jungen [1931]. That important paper[2] was partly motivated by Hadamard products and singular differential systems with regular singular points. There Jungen makes use of a method introduced by Fröbenius (and classical in the study of differential systems) that consists in starting from the binomial expansion (2.1) of $(1-z)^{\alpha}$ and differentiating with respect to the parameter α to deduce, with $\gamma = k$ an integer,

$$[z^n](1-z)^{\alpha} \log^k \frac{1}{1-z} \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \left[E_0(\log n) + \frac{E_1(\log n)}{n} + \frac{E_2(\log n)}{n^2} + \cdots \right],$$

where the E_j are polynomials of degree at most j. In other words, the expansion of our Theorem 3A terminates, and more terms in descending powers of n can be obtained. Another derivation, on which we partly based our proof of Theorem 3, is given by Flajolet and Puech [1986].

2. When α is a positive integer, $1/\Gamma(-\alpha) = 0$, so that the first term in the coefficient expansion of Theorem 3A vanishes and the expansion "jumps" to the next term in descending powers of $\log n$. For instance, we have

$$[z^n] \frac{1}{\log(1-z)^{-1}} \sim \frac{C}{n \log^2 n}.$$

Pólya cites without proof an example of this case in Pólya [1954, p. 9]. Pólya was also Jungen's advisor so that several of our theorems were probably known (or obvious) to him!

3. When both α and γ are positive integers, coefficients f_n are αth order differences of integral powers of a logarithm and explicit forms are directly available by the calculus of finite differences. For instance, with $\alpha = k$ (a positive integer) and $\gamma = 1$, we have

$$[z^n](1-z)^k \log \frac{1}{1-z} = (-1)^k \frac{k!}{n(n-1)\cdots(n-k)}.$$

In this context, we may refer to a short note by Zave [1976] that discusses the case where γ is an integer and α a *negative* integer. This is directly covered by our Theorem 3A, but Zave gives an interesting direct derivation using Bell polynomials and generalized harmonic numbers.

In other words, *the results of Theorem* 3A *remain valid when any of* α, γ *may be a positive integer provided we interpret* $1/\Gamma(-\alpha)$ *as* 0, *for* α *a nonnegative integer, and*

---

[2] Jungen proves the classical theorem: The *Hadamard* product of a *rational function* and an *algebraic function* is an *algebraic function*. It may be of interest to combinatorialists to note that this theorem has a multivariate noncommutative "lifting" due to Schützenberger, a special form of which is: *The intersection of a regular language and a context-free language is a context-free language*.

terminate descending expansions appropriately when terms become identically zero. We leave to the reader the pleasure of working out the degeneracy cases for Theorem 3B.

The first few terms of expansions in Theorems 3A and 3B are given below:

$$[z^n](1-z)^\alpha\left(\frac{1}{z}\log\frac{1}{1-z}\right)^\gamma = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\gamma\left[1-\frac{C_1}{1!}\frac{\gamma}{\log n}+\frac{C_2}{2!}\frac{\gamma(\gamma-1)}{(\log n)^2}+\cdots\right].$$

$$[z^n](1-z)^\alpha\left(\frac{1}{z}\log\frac{1}{1-z}\right)^\gamma\left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^\delta$$

$$= \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\gamma(\log\log n)^\delta\cdot\left[1-C_1\frac{(\delta+\gamma\log\log n)}{\log n\log\log n}\right.$$

$$\left.+\frac{C_2}{2}\frac{\delta(\delta-1)+\delta(2\gamma-1)\log\log n+\gamma(\gamma-1)(\log\log n)^2}{(\log n\log\log n)^2}+\cdots\right].$$

There $C_j = C_j(\alpha)$ represents

$$\Gamma(-\alpha)\frac{d^k}{ds^k}\frac{1}{\Gamma(-s)}\bigg|_{s=\alpha}.$$

4. Extensions: Large functions and slowly varying functions. We briefly give here indications on possible extensions of our methods, first to "large" functions, then to a set of slowly varying functions.

4.1. Large functions. For functions in class $\mathscr{S}$ that become "large" enough at their singularity, their dominant singular term being of the type $(1-z)^\alpha$ with $\alpha < -1$, the analyticity conditions of our previous theorems can be weakened. A corresponding form of Theorem 1 was given in Flajolet and Odlyzko [1982] (but we erroneously stated that form with the condition $\alpha < 0$ instead of the more restrictive $\alpha < -1$). For completeness, we state the corrected form and briefly sketch the proof.

THEOREM 4. *Assume that $f(z)$ is analytic in $|z| < 1$. Assume further that as $z \to 1$ in this domain,*

$$f(z) = O(|1-z|^\alpha), \tag{4.1a}$$

for some real number $\alpha < -1$. Then the nth Taylor coefficient of $f(z)$ satisfies

$$f_n = [z^n]f(z) = O(n^{-\alpha-1}). \tag{4.1b}$$

Proof. As our contour $\mathscr{C}$, we now choose

$$\mathscr{C} = \left\{z/|z| = 1-\frac{1}{n}\right\}.$$

For $z \in \mathscr{C}$, set $z = (1-(1/n))e^{i\theta}$, where $-\pi < \theta \leqq \pi$. Note that $|z|^{-n} \leqq 2e$ for $n \geqq 2$, and so

$$f_n = O\left(\int_0^\pi |1-z|^\alpha\, d\theta\right).$$

Now, for $\pi/2 \leqq \theta \leqq \pi$, $|1-z| > c$ for some constant $c > 0$, so that this region contributes a bounded quantity to f_n. We also have

$$\Re(1-z) = 1-\left(1-\frac{1}{n}\right)\cos\theta \geqq \frac{1}{n},$$

and, since $\sin\theta \geqq 2\theta/\pi$ for $0 \leqq \theta \leqq \pi/2$,

$$|\Im(1-z)| = \left(1-\frac{1}{n}\right)\sin\theta \geqq \frac{\theta}{10}.$$

Therefore, we get $|1-z| \geqq \frac{1}{2}((1/n)+(\theta/10))$, and

$$\int_0^{\pi/2} |1-z|^{\alpha}\,d\theta = O\left(\int_0^{\pi/2}\left(\frac{1}{n}+\frac{\theta}{10}\right)^{\alpha} d\theta\right)$$
$$= O(n^{-\alpha-1}),$$

which gives our estimate. □

Appropriate transfer results of all types will follow under the conditions of Theorem 4.

4.2. Slowly varying functions. We notice that, in the general proofs of § 3, it is possible to use, for all α, a contour $\mathscr{C}$ whose outer circle is of radius $R_n = 1 + \log^2 n/n$, so that R_n here plays the role of η. We could have used the same contour as in the proofs of § 3, but it is also instructive to introduce yet another contour.

It then becomes possible to extend the range of asymptotic scales leading to O-, o- and $\sim$-transfers to a scale that includes functions of the form $(1-z)^{\alpha}L((1-z)^{-1})$, provided $L(u)$ is of slow variation towards infinity. Such functions capture the features of functions like log, log log, etc. Function $L(u)$ is said to be of *slow variation* at ∞ if it satisfies the following conditions:

V1. There exists a positive real number u_0, and an angle ϕ with $0 < \phi < (\pi/2)$ such that $L(u)$ is $\neq 0$ and analytic in the domain

$$\{u/-(\pi-\phi) \leqq \operatorname{Arg}(u-u_0) \leqq (\pi-\phi)\}. \tag{4.2}$$

V2. There exists a function $\varepsilon(x)$, defined for $x \geqq 0$ with $\lim_{x\to+\infty}\varepsilon(x) = 0$, such that for all $\theta \in [-(\pi-\phi), \pi-\phi]$ and $u \geqq u_0$, we have

$$\left|\frac{L(ue^{i\theta})}{L(u)} - 1\right| < \varepsilon(u) \quad \text{and} \quad \left|\frac{L(u\log^2 u)}{L(u)} - 1\right| < \varepsilon(u). \tag{4.3}$$

THEOREM 5. *Assume that $L(u)$ is of slow variation at ∞, then the conditions*

$$f(z) = O\left((1-z)^{\alpha}L\left(\frac{1}{1-z}\right)\right),$$
$$f(z) = o\left((1-z)^{\alpha}L\left(\frac{1}{1-z}\right)\right), \quad f(z) \sim (1-z)^{\alpha}L\left(\frac{1}{1-z}\right),$$

as $z \to 1$ in $\Delta\setminus\{1\}$, transfer into the corresponding estimates for coefficients:

$$f_n = O(n^{-\alpha-1}L(n)), \quad f_n = o(n^{-\alpha-1}L(n)), \quad f_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}L(n).$$

Proof. (A simple adaptation of the proof of Corollary 1.) Inequalities (4.3) permit us to estimate the contributions to Cauchy's integral "near" the singularity $z = 1$, as though the $L((1-z)^{-1})$ terms were not present. □

This gives us a still wider range of functions such as

$$\exp(\sqrt{\log u}), \qquad \log\log\log u, \cdots.$$

For instance, we have the transfer

$$f(z)=O\left(\frac{1}{\sqrt{1-z}}\exp\left(\sqrt{\log\frac{1}{1-z}}\right)\right) \quad \Rightarrow \quad f_n=O(n^{-1/2}e^{\sqrt{\log n}}).$$

The slow variation conditions will just exclude a few functions that are very nearly a power of $(1-z)$, such as

$$\exp\left(\frac{\log(1-z)^{-1}}{\log\log(1-z)^{-1}}\right).$$

5. A comparison with alternative methods. It is interesting to note first that Darboux's method and Tauberian theorems are in a sense complementary. The former applies to functions of the form $(1-z)^\alpha$ where α is a sufficiently large positive number, while the latter necessitates $\alpha \leqq 0$. Transfer methods, which require somewhat different validity conditions, cover all values of α.

1. *Darboux's method.* There is a restricted form of Darboux's method which is most commonly encountered in combinatorial analysis. It applies to Taylor coefficients of functions of the form

$$f(z)=h(z)(1-z)^\alpha \qquad \alpha \notin \{0,1,2,\cdots\}, \tag{5.1}$$

where $h(z)$ is analytic in $|z|<1+\eta$ for some $\eta>0$. This is for instance the form given in Henrici [1977]. That form is directly covered by Corollary 4 and (2.27c).

The most general form is based on Darboux's lemma (a combination of integration by parts and the Riemann–Lebesgue lemma), also a classical result in Fourier analysis: *If $g(z)$ is analytic in $|z|<1$ and k times continuously differentiable on $|z|=1$, then*

$$[z^n]g(z)=o\left(\frac{1}{n^k}\right). \tag{5.2}$$

A typical application of the method to a function $f(x)$ therefore consists in finding a form

$$f(z)=\sigma(z)+g(z)$$

where $\sigma(z)$ is a simple singular function whose coefficients are known, and $g(z)$ is a remainder term that is amenable to treatment by Darboux's lemma.

We have already seen situations where the transfer approach applies while Darboux's method does not; this may owe to the very nature of the expansion (for example, (2.1a)), or to the fact that not enough terms can be obtained until a smooth enough error term (an instance is (2.28a)). Some further examples arising in applications are discussed in § 6 below. Conversely, it may be that a function is smooth on its circle of convergence, so that Darboux's method applies, but the circle is a natural boundary and no transfer like Theorem 1 is applicable. An artificial example[3] is

$$g(z)=\sin(z)\sum_{n=1}^{\infty}\frac{\lfloor\sqrt{n}\rfloor}{n^5}z^n. \tag{5.3}$$

In this case, $g(z)$ is three times continuously differentiable on $|z|=1$ and by Darboux, $g_n=o(n^{-3})$.

[3] The natural boundary property for $f(z)$ follows from the classical Pólya–Carlson theorem: *If a function is represented by a Taylor series that has integer coefficients and radius of convergence* 1, *either it is rational or it admits the unit circle as a natural boundary.*

In cases where both Darboux and transfer methods are applicable, transfer methods tend to give better estimates. For instance, if we determine the expansion of a function until a term of the form $g(z) = O((1-z)^{k+1/2})$ which is also k times continuously differentiable, we get $g_n = o(n^{-k})$ by Darboux, but a better bound of $O(n^{-k-3/2})$ by transfer.

2. *Tauberian theorems*. We already gave some indication in the introduction on this subject. In our terminology, a *real Tauberian theorem* asserts conditions under which an expansion

$$f(x) \sim \frac{c}{(1-x)^{\beta}}, \qquad (x \to 1^-)$$

(with $\beta \geqq 0$) that needs to be valid only along the *real* line, translates into an estimate for coefficients in the sense of a Cesarò average:

$$\frac{1}{n}\sum_{j=1}^{n} f_j \sim c\frac{n^{\beta-1}}{\Gamma(\beta)}.$$

A typical sufficient validity condition is $f_n > 0$. If, furthermore, the f_n are monotonic, then, we can infer that

$$f_n \sim c\frac{n^{\beta-1}}{\Gamma(\beta)}.$$

Application of Tauberian theorems therefore requires some a priori conditions—called Tauberian side conditions—like positivity, monotonicity, to be established on the coefficients.

Tauberian theorems are useful mostly for main terms in asymptotic expansions, and they may turn out to be the only applicable tool when the circle of convergence of the function is a natural boundary. Greene and Knuth [1983] have an interesting example of using a Tauberian theorem (complemented by "bootstrapping"), which gives the asymptotic form of coefficients of the function

$$f(z) = \prod_{k \geqq 1}\left(1 + \frac{z^j}{j}\right). \tag{5.4}$$

Function f has the unit circle as a natural boundary and does not seem amenable to transfer methods.

Conversely, a function with somewhat erratic coefficients such as

$$\sin z \cos\left(\log\frac{1}{1-z}\right)$$

is easily treated by transfer methods (using transfers $(1-z)^{\pm i}$), but the coefficients are not smooth enough to allow application of a Tauberian theorem.

On all those classical questions the reader is encouraged to refer to many excellent books like De Bruijn [1981], Olver [1974], Henrici [1977], and, for Tauberian theory, Hardy [1949] or Postnikov [1980].

6. Applications. In this section, we propose to review a few applications of transfer methods in combinatorial analysis and analysis of algorithms.

2-3 trees. The problem dealt with in Odlyzko [1982] was at the origin of our interest in transfer methods. It consisted of determining the number of balanced 2-3 trees with

n external nodes. This reduces to the determination of the coefficients of a generating function $f(z)$ that satisfies the functional equation

$$f(z)=z+f(z^2+z^3). \tag{6.1}$$

It can be recognized that $f(z)$ is analytic for $|z| < \phi^{-1}$ where $\phi = (1+\sqrt{5})/2$ is the golden ratio (ϕ^{-1} is a fixed point of z^2+z^3). A detailed study of the iteration of polynomial $\sigma(z) = z^2 + z^3$ shows that there is a singular expansion at $z = \phi^{-1}$ which is of the form

$$\begin{aligned} f(z) &= c \log \frac{1}{1-\phi z} + \sum_{k=-\infty}^{k=+\infty} \Omega_k (1-z)^{2ik\pi/\lambda} + O((1-\phi z)) \\ &= c \log \frac{1}{1-\phi z} + \Omega\left(\frac{1}{\lambda} \log(1-\phi z)\right) + O((1-\phi z)). \end{aligned} \tag{6.2a}$$

There $\lambda = \log(4-\phi)$ and the infinite sum $\Omega(x)$ in (6.2a) is a fast converging Fourier series with period 1. That singular expansion can be established in an angular sector around the singularity ϕ^{-1}, so that expansion (6.2a) transfers to coefficients,

$$\begin{aligned} \phi^{-n} f_n &= \frac{c}{n} + \sum_{k=-\infty}^{+\infty} \omega_k n^{-1-2ik\pi/\lambda} + O\left(\frac{1}{n^2}\right) \\ &= \frac{c}{n} + \frac{1}{n}\omega\left(\frac{1}{\lambda}\log n\right) + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{6.2b}$$

where $\omega(x)$ is a Fourier series (with period 1 and mean value 0) that "corresponds" to $\Omega(x)$. Form (6.2b) gives the asymptotic number of 2-3 trees with n external nodes.

This example illustrates a direct extension of Corollary 3 to the situation where the singular expansion (6.2a) contains infinitely many terms whose complex exponents have a common real part.

Height of trees. The problem of estimating the expected height of a binary tree with n internal nodes (Flajolet and Odlyzko [1982]) reduces easily to finding an asymptotic expansion for the coefficients of function $f(z)$ given by

$$\begin{aligned} &f(z) = \sum_{h \geq 0} [y(z) - y_h(z)] \\ &\text{where } y_0(z) = 0;\ y_{h+1}(z) = 1 + z y_h(z);\ y(z) = y_\infty(z) = \frac{1-\sqrt{1-4z}}{2z}. \end{aligned} \tag{6.3}$$

There is a somewhat delicate analysis to determine the behaviour of the quadratic recurrence at $z = \frac{1}{4}$, from which we obtain

$$f(z) = c \log \frac{1}{1-4z} + c_0 + O((1-4z)^{1/4+\varepsilon}), \tag{6.4a}$$

for any $\varepsilon > 0$. By a direct application of Corollary 3, we derive

$$4^{-n} f_n = \frac{c}{n} + O(n^{-5/4+\varepsilon}). \tag{6.4b}$$

From (6.4b), we find after normalization that the expected height of a binary tree with n internal nodes is $2\sqrt{\pi n} + O(n^{1/4+\varepsilon})$. Here transfer lemmas are useful since it seems difficult to obtain more terms in expansion (6.4a), and application of Darboux's method (if at all feasible) would have required an expansion until terms of a higher order like

$O((1-4z)^{3/2})$. Computation of higher moments led us to develop the contour used in Theorem 4.

Multidimensional search. The analysis of partial match retrieval (Flajolet and Puech [1986]) in so-called k-d-trees requires expanding a function $f(z)$ that is a component of the solution to a linear differential system with regular singular points. Using the standard theory of regular singular points, we find a singular expansion

$$f(z)=c(1-z)^{-\beta}+O\left(\frac{1}{(1-z)^2}\right) \tag{6.5a}$$

valid at an angle outside the circle of convergence $|z|=1$, where $2<\beta<3$. (For a 2-d search, $\beta=(\sqrt{17}+1)/2$.) It would have been possible (though a little more lengthy) to push expansion (6.5a) further, but Theorem 1 or Theorem 4 provide all that is needed to deduce directly

$$f_n=\frac{c}{\Gamma(\beta)}n^{\beta-1}+O(n). \tag{6.5b}$$

For instance, a partial match search in a 2-d tree will have expected cost $\sim Kn^{(\sqrt{17}-3)/2}$.

Common subexpression problem. The analysis of the representation of trees by compact directed acyclic graphs (dags) in Flajolet, Sippala, and Steyaert [1987] requires finding the coefficient of z^n in

$$f(z)=\frac{1}{2z}\sum_{p\geq 0}B_p[\sqrt{1-4z+4z^{p+1}}-\sqrt{1-4z}] \quad \text{where } B_p=\frac{1}{p+1}\binom{2p}{p}. \tag{6.6}$$

A singularity analysis of (6.6) around $z=\frac{1}{4}$ shows that

$$f(z)=\frac{c}{\sqrt{(1-4z)\log(1-4z)^{-1}}}\left[1+O\left(\frac{1}{\log(1-4z)^{-1}}\right)\right]. \tag{6.7a}$$

By Theorem 3A and a trivially amended form of Corollary 5, we get

$$4^{-n}f_n=\frac{c}{\sqrt{\pi n\log n}}\left[1+O\left(\frac{1}{\log n}\right)\right]. \tag{6.7b}$$

This example (see also the discussion of (1.2a), (1.2b)) is interesting since it could not be attacked by Darboux's method. As we have already indicated, even by pushing the expansion further, there is no way of obtaining a differentiable error term in a more extensive form of (6.7a). Tauberian methods would be possible candidates for attacking (6.7a), but it seems to be quite difficult to establish Tauberian side conditions on the *error term* in (6.7a), and the situation would get even worse if higher order terms were to be found. By transfer methods, we are able to conclude easily that the expected size of the maximally compacted dag representation of a tree of size n is $\sim cn/\sqrt{\log n}$.

Longest cycle in permutations. This problem, solved by Shepp and Lloyd [1966], is equivalent to finding the asymptotic form of the coefficients of

(6.8)

$$f(z)=\sum_{k\geq 0}\left[\frac{1}{1-z}-\exp(z/1+z^2/2+\cdots+z^k/k)\right]=\frac{1}{1-z}\sum_{k\geq 0}\left[1-\exp\left(-\sum_{j\geq k}\frac{z^j}{j}\right)\right].$$

Function $f(z)$ is singular at $z = 1$, and it is natural to set $z = e^{-t}$ so that $t \to 0$ as $z \to 1$ and $z \sim 1 - t$. Two successive applications of the Euler–Maclaurin summation to (6.8) provide the approximation

$$(6.9a) \qquad (1-z)f(z) \sim \frac{G}{t} \quad \text{where } G = \int_0^\infty \left[1 - \exp\left(-\int_x^\infty \frac{e^{-t}}{t}\right) dt\right] dx.$$

Thus $F(z) \sim G(1-z)^{-2}$, and by transfer (Corollary 3), the longest cycle in a random permutation of n elements has expected length

$$(6.9b) \qquad f_n \sim Gn.$$

We observe that Shepp and Lloyd's original derivation (see also Knuth [1973a, p. 181]) proceeds along quite different lines. They first prove a Poisson approximation and then use a Tauberian theorem. But Tauberian side conditions, though expected combinatorially, are not obvious to establish.

In Flajolet and Odlyzko [1990], we apply an analysis of this type to study random mappings and find the expected diameter of a random mapping of size n.

Odd-even merge. The problem of analyzing odd-even merge sorting networks was posed by Knuth [1973b, Ex. 5.2.2.16]. Knuth reduces it to finding the Taylor coefficients of a function closely related to

$$(6.10) \qquad f(z) = \frac{y}{1+y^2} + \frac{y^2}{1+y^4} + \frac{y^4}{1+y^8} + \cdots \quad \text{where } y = \frac{1-\sqrt{1-4z}}{1+\sqrt{1-4z}}.$$

The problem was solved by Sedgewick in 1977 by expanding, then using real approximations on coefficients and finally applying Mellin transform techniques (Sedgewick [1977]). We present here the outline of an alternative approach based on Flajolet and Prodinger [1986]. Following our general strategy, we try to determine a singular expansion of $f(z)$ around the singularity $z = \frac{1}{4}$. We can set $y(z) = e^{-t}$, so that $t \to 0$ as $z \to \frac{1}{4}$, and the problem is to analyze

$$(6.11) \qquad F(t) = \frac{e^{-t}}{1+e^{-2t}} + \frac{e^{-2t}}{1+e^{-4t}} + \frac{e^{-4t}}{1+e^{-8t}} + \cdots \quad \text{as } (t \to 0).$$

We assume, to simplify the discussion, that t real and positive (general theorems guarantee that the expansion can be "continued" for complex t with $\Re(t) > 0$, hence for complex z at an angle from $\frac{1}{4}$). The Mellin transform of $F(t)$ is easily found from (6.11),

$$(6.12) \quad F^*(s) \overset{\text{def}}{\equiv} \int_0^\infty F(t) t^{s-1}\, dt = \frac{\Gamma(s)L(s)}{1-2^{-s}} \quad \text{where } L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots.$$

From the familiar Mellin inversion formula and a computation by residues (Doetsch [1955]), we obtain an asymptotic expansion of $F(t)$ as $t \to 0$,

$$(6.13) \qquad F(t) \sim \sum \operatorname{Res}\,[F^*(s)t^{-s}],$$

where the sum is extended to all poles s of $F^*(s)$ satisfying $\Re(s) \leqq 0$: There is a double pole at $s = 0$, simple imaginary poles at $s = 2ik\pi/\log 2$, and simple real poles at $s = -2, -4, -6, \cdots$. Thus,

$$(6.14a) \qquad F(t) \sim \frac{1}{2}\log_2\frac{1}{t} + c_0 + \Omega(\log_2 t) + 2\sum_{k \geqq 1} \frac{E_{2k}}{1-2^{-k}} \frac{t^{2k}}{((2k)!)^2}.$$

There $\Omega(x)$ is a Fourier series in x and E_{2n} is the Euler number,

$$E_{2n}=(2n)![z^{2n}](1/\cos z).$$

Expansion (6.14a) is a *full* expansion in increasing powers of t which, by transformation $t=-\log y(z)$, yields a full asymptotic expansion of $F(t)$. From there, we can find a full expansion for the coefficients of $f(z)$ (by transfer) and, in particular, get a complete asymptotic expansion of f_n. The same method gives Sedgewick's result (and an infinite expansion): *The expected number of exchanges in odd-even merge applied to two sequences of length n is $\frac{1}{4}n\log_2 n+\cdots$*. We refer the reader to Flajolet and Prodinger [1986] for a similar example treated in detail.

This "synthetic" method differs from the usual approach of De Bruijn, Knuth, and Rice [1972]: In accordance with our general principles, we operated only at the level of the generating function (using complex Mellin transforms to derive a singular expansion), and concluded directly by a transfer theorem.

Limit distributions in combinatorics. Consider a bivariate generating function of the form

$$P(z,u)=\exp(uG(z)).$$

Such functions arise naturally in counting combinatorial structures $\mathscr{P}$ that are decomposable as sets of basic building blocks (components) enumerated by $G(z)$. In this context, the polynomials $p_n(u)=[z^n]P(z,u)$ are the generating polynomials giving the distribution of the number of components in a random $\mathscr{P}$ structure of size n. Under the condition that $G(z)$ has a dominant singularity of a logarithmic type, methods of this paper may be used to estimate asymptotically the characteristic function $p_n(e^{it})$ which, once suitably normalised, tends to $e^{-t^2/2}$. From the continuity theorem for characteristic functions, we deduce that the number of components in a "large" ($n\to\infty$) random $\mathscr{P}$ structure tends to a limiting Gaussian distribution. A typical application is to cycles in permutations. There are extensions to Pólya's theory of counting, with the corresponding analytic scheme

$$P(z,u)=\exp\left(\frac{u}{1}G(z)\pm\frac{u^2}{2}G(z^2)+\frac{u^3}{3}G(z^3)\pm\frac{u^4}{4}G(z^4)+\cdots\right).$$

We obtain, for instance: *The number of irreducible factors in a random polynomial with coefficients in $GF(q)$ is asymptotically Gaussian.*

Under different analytic conditions on the "generator" $G(z)$, Bender [1973] and Canfield [1977] have established similar asymptotic normality results. Detailed proofs of results cited here are presented in Flajolet and Soria [1988].

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