SHI Hyperbolic Sine Integral

SHI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Hyperbolic Sine Integral (noted Shi) is defined by the following third order differential equation

$$-x\frac{\partial y(x)}{\partial x} + 2\frac{\partial^2 y(x)}{\partial x^2} + x\frac{\partial^3 y(x)}{\partial x^3} = 0. \tag{SHI.1.1}$$

Although 0 is a singularity of SHI.1.1, the initial conditions can be given by

$$\frac{\partial \frac{\mathrm{Shi}(x)}{x}}{\partial x} = 1. \tag{SHI.1.2}$$

Related function: Hyperbolic Cosine Integral

SHI.2 Series and asymptotic expansions

SHI.2.1 Asymptotic expansion at ∞.

SHI.2.1.1 First terms.

$$\mathrm{Shi}(x) \approx$$
$$ser_{\left[1,1,\left[\left[0,\left[\left[0,\frac{\pi}{2}\right]\right]\right]\right]\right]} + \mathrm{e}^{\frac{1}{x}}\, x y_1(x) + \mathrm{e}^{\left(-\frac{1}{x}\right)}\, x y_2(x),$$

where

$$y_0(x) = terms_{\left[1,1,\left[\left[0,\left[\left[0,\frac{\pi}{2}\right]\right]\right]\right]\right]} + \ldots$$
$$y_1(x) = \frac{1}{2} + \frac{x}{2} + x^2 + 3x^3 + 2\ldots$$
$$y_2(x) = \frac{1}{2} - \frac{x}{2} + x^2 - 3x^3 + 2\ldots$$

SHI.2.1.2 General form.

SHI.2.1.2.1 Auxiliary function $y_0(x)$. The auxiliary function $y_0(x)$ has the exact form

$$y_0(x) = \frac{\pi}{2}$$

SHI.2.1.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$-2u(n)n + u(n-1)\big(-3 + 3(n-1)^2 + 5n\big) +$$
$$u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(1) = \frac{1}{2}$$
$$u(0) = \frac{1}{2}$$

This recurrence has the closed form solution

$$u(n) = \frac{\Gamma(n+1)}{2}.$$

SHI.2.1.2.3 Auxiliary function $y_2(x)$. The coefficients $u(n)$ of $y_2(x)$ satisfy the following recurrence

$$-2u(n)n + u(n-1)\big(3 - 3(n-1)^2 - 5n\big) + u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(0) = \frac{1}{2}$$
$$u(1) = -\frac{1}{2}$$

This recurrence has the closed form solution

$$u(n) = \frac{(-1)^n \Gamma(n+1)}{2}.$$

SHI.2.2 Asymptotic expansion at 0.

SHI.2.2.1 First terms.

(SHI.2.2.1.1)
$$\mathrm{Shi}(x) \approx x\left(1 + \frac{x^2}{18} + \frac{x^4}{600} + \frac{x^6}{35280} + \frac{x^8}{3265920} + \frac{x^{10}}{439084800} + \frac{x^{12}}{80951270400} + \frac{x^{14}}{19615115520000} \dots\right).$$

SHI.2.2.2 General form.

(SHI.2.2.2.1)
$$\mathrm{Shi}(x) \approx x \sum_{n=0}^{\infty} u(n) x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(SHI.2.2.2.2)
$$-u(n)(n+1)\big(1 + n - (n+1)^2\big) + u(n-2)(1-n) = 0.$$

Initial conditions of SHI.2.2.2.2 are given by

(SHI.2.2.2.3)
$$u(1) = 0, \quad u(0) = 1, \quad u(2) = \frac{1}{18}.$$

The recurrence SHI.2.2.2.2 has the closed form solution

(SHI.2.2.2.4)
$$u(2n+1) = 0, \quad u(2n) = \frac{1}{(2n+1)\Gamma(2n+2)}.$$