HATN Inverse Hyperbolic Tangent

HATN.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{-1, 1\}$.The function Inverse Hyperbolic Tangent (noted arctanh) is defined by the following second order differential equation

$$2x\frac{\partial y(x)}{\partial x} + \left(x^2 - 1\right)\frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{HATN.1.1}$$

The initial conditions of HATN.1.1 are given at 0 by

$$\begin{aligned} \operatorname{arctanh}(0) &= 0, \\ \frac{\partial \operatorname{arctanh}(x)}{\partial x}(0) &= 1. \end{aligned} \tag{HATN.1.2}$$

Related function: Inverse Hyperbolic Cotangent

HATN.2 Series and asymptotic expansions

HATN.2.1 Taylor expansion at 0.

HATN.2.1.1 First terms.

$$\begin{aligned} \operatorname{arctanh}(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} + \frac{1}{13}x^{13} + \frac{1}{15}x^{15} \\ + \mathrm{O}\left(x^{16}\right). \end{aligned} \tag{HATN.2.1.1.1}$$

HATN.2.1.2 General form.

$$\operatorname{arctanh}(x) = \sum_{n=0}^{\infty} u(n)x^n. \tag{HATN.2.1.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$nu(n) - -(-n-2)u(n+2) = 0. \tag{HATN.2.1.2.2}$$

Initial conditions of HATN.2.1.2.2 are given by

$$\begin{aligned} u(1) &= 1, \\ u(0) &= 0. \end{aligned} \tag{HATN.2.1.2.3}$$

HATN.2.2 Asymptotic expansion at 1.

HATN.2.2.1 First terms.

$$\begin{aligned} \operatorname{arctanh}(x) \approx \left(\frac{i}{2}\pi + \frac{\ln(2)}{2} - \frac{x}{4} + \frac{1}{4} + \frac{(x-1)^2}{16} - \frac{(x-1)^3}{48} + \frac{(x-1)^4}{128} - \right. \\ \left. \frac{(x-1)^5}{320} + \frac{(x-1)^6}{768} - \frac{(x-1)^7}{1792} + \frac{(x-1)^8}{4096} + \frac{\ln(x-1)}{2} \dots\right). \end{aligned} \tag{HATN.2.2.1.1}$$

HATN.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

HATN.2.3 Asymptotic expansion at -1.

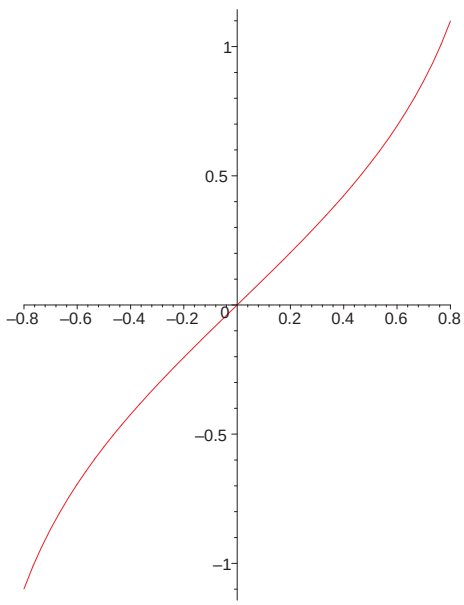
HATN.2.3.1 First terms.

$$\operatorname{arctanh}(x) \approx \left(\frac{-\ln(2)}{2} - \frac{x}{4} - \frac{1}{4} - \frac{(x+1)^2}{16} - \frac{(x+1)^3}{48} - \frac{(x+1)^4}{128} - \frac{(x+1)^5}{320} - \frac{(x+1)^6}{768} - \frac{(x+1)^7}{1792} - \frac{(x+1)^8}{4096} - \frac{\ln(x+1)}{2} \cdots \right). \tag{HATN.2.3.1.1}$$

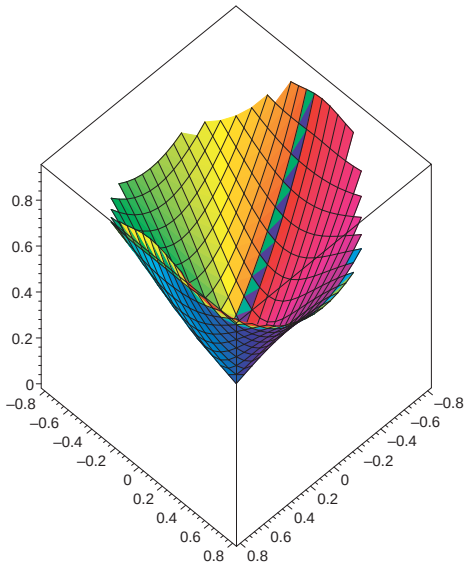
HATN.2.3.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

HATN.3 Graphs

HATN.3.1 Real axis.



HATN.3.2 Complex plane.


0.8
0.6
0.4
0.2
0
–0.8
–0.6
–0.4
–0.2
0
0.2
0.4
0.6
0.8
–0.8
–0.6
–0.4
–0.2
0
0.2
0.4
0.6
0.8