ERF Error Function

ERF.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Error Function (noted erf) is defined by the following second order differential equation

(ERF.1.1) $$2x\frac{\partial y(x)}{\partial x} + \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

The initial conditions of ERF.1.1 are given at 0 by

(ERF.1.2) $$\operatorname{erf}(0) = 0, \quad \frac{\partial \operatorname{erf}(x)}{\partial x}(0) = \frac{2}{\sqrt{\pi}}.$$

Related function: Complementary Error Function

ERF.2 Series and asymptotic expansions

ERF.2.1 Taylor expansion at 0.

ERF.2.1.1 First terms.

(ERF.2.1.1.1) $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 - \frac{1}{21\sqrt{\pi}}x^7 + \frac{1}{108\sqrt{\pi}}x^9 - \frac{1}{660\sqrt{\pi}}x^{11} + \frac{1}{4680\sqrt{\pi}}x^{13} - \frac{1}{37800\sqrt{\pi}}x^{15} + \mathrm{O}\big(x^{16}\big).$$

ERF.2.1.2 General form.

(ERF.2.1.2.1) $$\operatorname{erf}(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(ERF.2.1.2.2) $$2nu(n) + \big(n^2 + 3n + 2\big)u(n+2) = 0.$$

Initial conditions of ERF.2.1.2.2 are given by

(ERF.2.1.2.3) $$u(0) = 0, \quad u(1) = \frac{2}{\sqrt{\pi}}.$$

ERF.2.2 Asymptotic expansion at ∞.

ERF.2.2.1 First terms.

$$\operatorname{erf}(x) \approx \mathrm{e}^{\left(-\frac{1}{x^2}\right)} x y_0(x) + ser_{\left[1,1,\left[\left[0,\left[[0,1]\right]\right]\right]\right]},$$

where

$$y_0(x) = -\frac{1}{\sqrt{\pi}} + \frac{x^2}{2\sqrt{\pi}} + 2\dots$$
$$y_1(x) = terms_{\left[1,1,\left[\left[0,\left[[0,1]\right]\right]\right]\right]} + \dots$$

ERF.2.2.2 General form.

ERF.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$2nu(n) + u(n-2)\left(-4 + 3n + (n-2)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = -\frac{1}{\sqrt{\pi}}$$
$$u(1) = 0$$

This recurrence has the closed form solution

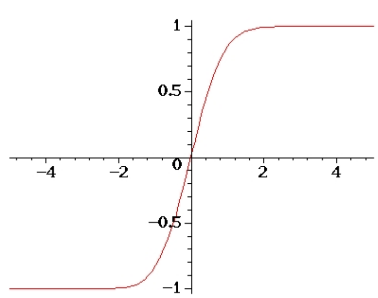
$$u(2n+1) = 0,$$
$$u(2n) = -\frac{(-1)^n \Gamma\left(n + \frac{1}{2}\right)}{\pi}.$$

ERF.2.2.2.2 Auxiliary function $y_1(x)$. The auxiliary function $y_1(x)$ has the exact form

$$y_1(x) = 1$$

ERF.3 Graphs

ERF.3.1 Real axis.



ERF.3.2 Complex plane.