DBI Derivative of Airy Bi

DBI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Derivative of Airy Bi (noted Bi$\prime$) is defined by the following second order differential equation

(DBI.1.1) $$-x^2 y(x) - \frac{\partial y(x)}{\partial x} + x \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

Although 0 is a singularity of DBI.1.1, the initial conditions can be given by

(DBI.1.2) $$[1] \operatorname{Bi}\prime(x) = \frac{3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{2\pi}, \qquad \left[x^2\right] \operatorname{Bi}\prime(x) = \frac{3^{\frac{5}{6}}}{6\Gamma\left(\frac{2}{3}\right)}.$$

Related function: Derivative of Airy Ai

DBI.2 Series and asymptotic expansions

DBI.2.1 Asymptotic expansion at ∞.

DBI.2.1.1 First terms.

$$\operatorname{Bi}\prime(x) \approx \frac{e^{\left(\frac{-2}{3\xi^3}\right)} \left(\frac{i}{\sqrt{\pi}} + \frac{7i\xi^3}{48\sqrt{\pi}} + \dots\right)}{\sqrt{\xi}}$$

where $\xi = -\sqrt{\frac{1}{x}}$

DBI.2.1.2 General form.

$$\operatorname{Bi}\prime(x) \approx \frac{e^{\left(\frac{-2}{3\xi^3}\right)} \sum_{n=0}^{\infty} u(n)\xi^n}{\sqrt{\xi}}$$

where $\xi = -\sqrt{\frac{1}{x}}$The coefficients $u(n)$ satisfy the following recurrence

$$16u(n)n + u(n-3)\left(-43 + 12n + 4(n-3)^2\right) = 0.$$

whose initial conditions are given by

$$u(0) = \frac{i}{\sqrt{\pi}}, \quad u(2) = 0, \quad u(1) = 0.$$

This recurrence has the closed form solution

$$\begin{aligned} u(3n+1) &= 0, \\ u(3n+2) &= 0, \\ u(3n) &= \frac{-i(-1)^n 6^{(2n)} \Gamma\left(n+\frac{7}{6}\right) \Gamma\left(n-\frac{1}{6}\right)}{2\pi^{\frac{3}{2}} 48^n \Gamma(n+1)}. \end{aligned}$$

DBI.2.2 Asymptotic expansion at 0.

DBI.2.2.1 First terms.

$$\operatorname{Bi}\prime(x) \approx \left(\frac{3^{\frac{5}{6}} x^8}{4320\Gamma\left(\frac{2}{3}\right)} + \frac{x^6 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{144\pi} + \frac{3^{\frac{5}{6}} x^5}{90\Gamma\left(\frac{2}{3}\right)} + \frac{x^3 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{6\pi} + \frac{3^{\frac{5}{6}} x^2}{6\Gamma\left(\frac{2}{3}\right)} + \frac{3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{2\pi} \cdots\right). \tag{DBI.2.2.1.1}$$

DBI.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).