

# DAI Derivative of Airy Ai

## DAI.1 Introduction

Let  $x$  be a complex variable of  $\mathbb{C} \setminus \{\infty\}$ . The function Derivative of Airy Ai (noted  $\text{Ai}'$ ) is defined by the following second order differential equation

$$(DAI.1.1) \quad -x^2 y(x) - \frac{\partial y(x)}{\partial x} + x \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

Although 0 is a singularity of DAI.1.1, the initial conditions can be given by

$$(DAI.1.2) \quad \begin{aligned} [1] \text{Ai}'(x) &= \frac{-\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{2\pi}, \\ [x^2] \text{Ai}'(x) &= \frac{\sqrt[3]{3}}{6\Gamma\left(\frac{2}{3}\right)}. \end{aligned}$$

Related function: Derivative of Airy Bi

## DAI.2 Series and asymptotic expansions

### DAI.2.1 Asymptotic expansion at $\infty$ .

*DAI.2.1.1 First terms.*

$$\text{Ai}'(x) \approx \frac{e^{\left(\frac{-2}{3\xi^3}\right)} \left( \frac{-1}{2\sqrt{\pi}} - \frac{7\xi^3}{96\sqrt{\pi}} + \dots \right)}{\sqrt{\xi}}$$

where  $\xi = \sqrt{\frac{1}{x}}$

*DAI.2.1.2 General form.*

$$\text{Ai}'(x) \approx \frac{e^{\left(\frac{-2}{3\xi^3}\right)} \sum_{n=0}^{\infty} u(n)\xi^n}{\sqrt{\xi}}$$

where  $\xi = \sqrt{\frac{1}{x}}$  The coefficients  $u(n)$  satisfy the following recurrence

$$16u(n)n + u(n-3)(-43 + 12n + 4(n-3)^2) = 0.$$

whose initial conditions are given by

$$\begin{aligned} u(0) &= \frac{-1}{2\sqrt{\pi}}, \\ u(1) &= 0, \\ u(2) &= 0. \end{aligned}$$

This recurrence has the closed form solution

$$u(3n+1) = 0,$$

$$u(3n+2) = 0,$$

$$u(3n) = \frac{(-1)^n 6^{(2n)} \Gamma\left(n + \frac{7}{6}\right) \Gamma\left(n - \frac{1}{6}\right)}{4\pi^{\frac{3}{2}} 48^n \Gamma(n+1)}.$$

### DAI.2.2 Asymptotic expansion at 0.

*DAI.2.2.1 First terms.*

$$(DAI.2.2.1.1) \quad \text{Ai}'(x) \approx \left( \frac{\sqrt[3]{3}x^8}{4320\Gamma\left(\frac{2}{3}\right)} - \frac{x^6 \sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{144\pi} + \frac{\sqrt[3]{3}x^5}{90\Gamma\left(\frac{2}{3}\right)} - \frac{x^3 \sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{6\pi} + \frac{\sqrt[3]{3}x^2}{6\Gamma\left(\frac{2}{3}\right)} - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{2\pi} \dots \right).$$

*DAI.2.2.2 General form.* The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).