CHI Hyperbolic Cosine Integral

CHI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$.The function Hyperbolic Cosine Integral (noted Chi) is defined by the following third order differential equation

$$-x\frac{\partial y(x)}{\partial x} + 2\frac{\partial^2 y(x)}{\partial x^2} + x\frac{\partial^3 y(x)}{\partial x^3} = 0. \tag{CHI.1.1}$$

The initial conditions of CHI.1.1 at 0 are not simple to state, since 0 is a (regular) singular point.

Related function: Hyperbolic Sine Integral

CHI.2 Series and asymptotic expansions

CHI.2.1 Asymptotic expansion at 0.

CHI.2.1.1 First terms.

$$\mathrm{Chi}(x) \approx \left(\frac{-x^2}{4} - \frac{x^4}{96} - \frac{x^6}{4320} - \frac{x^8}{322560} - \ln(x) + \gamma \dots\right). \tag{CHI.2.1.1.1}$$

CHI.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

CHI.2.2 Asymptotic expansion at ∞.

CHI.2.2.1 First terms.

$$\mathrm{Chi}(x) \approx \mathrm{e}^{\frac{1}{x}}\, x y_0(x) + \mathrm{e}^{\left(-\frac{1}{x}\right)}\, x y_1(x),$$

where

$$y_0(x) = \frac{1}{2} + \frac{x}{2} + x^2 + 3x^3 + 2\dots$$
$$y_1(x) = -\frac{1}{2} + \frac{x}{2} - x^2 + 3x^3 + 2\dots$$

CHI.2.2.2 General form.

CHI.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$-2u(n)n + u(n-1)\big(-3 + 3(n-1)^2 + 5n\big) +$$
$$u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(1) = \frac{1}{2}$$
$$u(0) = \frac{1}{2}$$

This recurrence has the closed form solution

$$u(n) = \frac{\Gamma(n+1)}{2}.$$

CHI.2.2.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$\begin{gathered}
-2u(n)n + u(n-1)\big(3 - 3(n-1)^2 - 5n\big) + \\
u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0
\end{gathered}$$

whose initial conditions are given by

$$\begin{aligned}
u(1) &= \frac{1}{2} \\
u(0) &= -\frac{1}{2}
\end{aligned}$$

This recurrence has the closed form solution

$$u(n) = \frac{-(-1)^n \Gamma(n+1)}{2}.$$