ATN Inverse Tangent

ATN.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{i, -i\}$.The function Inverse Tangent (noted arctan) is defined by the following second order differential equation

$$-2x\frac{\partial y(x)}{\partial x} - -\left(-1 - x^2\right)\frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{ATN.1.1}$$

The initial conditions of ATN.1.1 are given at 0 by

$$\begin{aligned} \arctan(0) &= 0, \\ \frac{\partial \arctan(x)}{\partial x}(0) &= 1. \end{aligned} \tag{ATN.1.2}$$

ATN.2 Series and asymptotic expansions

ATN.2.1 Asymptotic expansion at i.

ATN.2.1.1 First terms.

$$\begin{aligned} \arctan(x) \approx \Bigg(& \frac{i}{2}\ln(2) + \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)}{4} \\ & - \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^2}{16} + \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^3}{48\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)} - \\ & \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^4}{128\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2} + \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^5}{320\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^3} - \\ & \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^6}{768\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^4} + \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^7}{1792\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^5} - \\ & \frac{\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^8}{4096\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^6} + \frac{\ln\left(x - \mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)}{2}\cdots\Bigg). \end{aligned} \tag{ATN.2.1.1.1}$$

ATN.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

ATN.2.2 Asymptotic expansion at $-i$.

ATN.2.2.1 First terms.

$$\begin{aligned}\arctan(x) \approx \Bigg(& -\frac{i}{2}\ln(2) + \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)}{4} \\ & - \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^2}{16} + \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^3}{48\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)} - \\ & \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^4}{128\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2} + \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^5}{320\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^3} - \\ & \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^6}{768\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^4} + \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^7}{1792\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^5} - \\ & \frac{\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^8}{4096\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^6} + \frac{\ln\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)}{2}\cdots\Bigg).\end{aligned} \tag{ATN.2.2.1.1}$$

ATN.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

ATN.2.3 Taylor expansion at 0.

ATN.2.3.1 First terms.

$$\begin{aligned}\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} - \frac{1}{15}x^{15} \\ + \mathrm{O}\left(x^{16}\right).\end{aligned} \tag{ATN.2.3.1.1}$$

ATN.2.3.2 General form.

$$\arctan(x) = \sum_{n=0}^{\infty} u(n)x^n. \tag{ATN.2.3.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

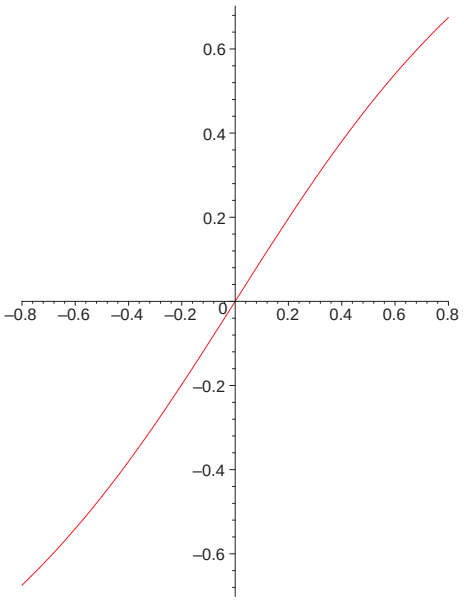
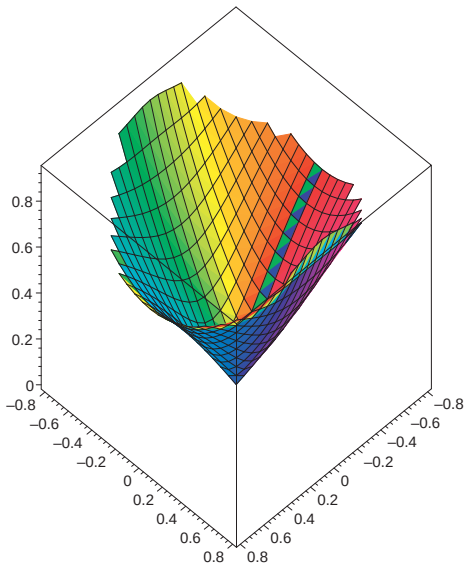
$$-nu(n) - -(-n-2)u(n+2) = 0. \tag{ATN.2.3.2.2}$$

Initial conditions of ATN.2.3.2.2 are given by

$$\begin{aligned}u(0) &= 0, \\ u(1) &= 1.\end{aligned} \tag{ATN.2.3.2.3}$$

ATN.3 Graphs

ATN.3.1 Real axis.



ATN.3.2 Complex plane.