AI Airy Ai

AI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Airy Ai (noted Ai) is defined by the following second order differential equation

(AI.1.1)
$$-xy(x) + \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

The initial conditions of AI.1.1 are given at 0 by

(AI.1.2)
$$\operatorname{Ai}(0) = \frac{\sqrt[3]{3}}{3\Gamma\left(\frac{2}{3}\right)},$$
$$\frac{\partial \operatorname{Ai}(x)}{\partial x}(0) = \frac{-\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{2\pi}.$$

Related function: Airy Bi

AI.2 Series and asymptotic expansions

AI.2.1 Taylor expansion at 0.

AI.2.1.1 First terms.

(AI.2.1.1.1)
$$\operatorname{Ai}(x) = \frac{\sqrt[3]{3}}{3\Gamma\left(\frac{2}{3}\right)} - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{2\pi}x + \frac{\sqrt[3]{3}}{18\Gamma\left(\frac{2}{3}\right)}x^3 - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{24\pi}x^4 + \frac{\sqrt[3]{3}}{540\Gamma\left(\frac{2}{3}\right)}x^6 - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{1008\pi}x^7 + \frac{\sqrt[3]{3}}{38880\Gamma\left(\frac{2}{3}\right)}x^9 - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{90720\pi}x^{10} + \frac{\sqrt[3]{3}}{5132160\Gamma\left(\frac{2}{3}\right)}x^{12} - \frac{\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{14152320\pi}x^{13} + \frac{\sqrt[3]{3}}{1077753600\Gamma\left(\frac{2}{3}\right)}x^{15} + \mathrm{O}\left(x^{16}\right).$$

AI.2.1.2 General form.

(AI.2.1.2.1)
$$\operatorname{Ai}(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(AI.2.1.2.2)
$$-u(n) + \left(n^2 + 5n + 6\right)u(n+3) = 0.$$

Initial conditions of AI.2.1.2.2 are given by

$$
\begin{aligned}
u(0) &= \frac{\sqrt[3]{3}}{3\Gamma\left(\frac{2}{3}\right)}, \\
u(1) &= \frac{-\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)}{2\pi}, \\
u(2) &= 0.
\end{aligned} \tag{AI.2.1.2.3}
$$

The recurrence AI.2.1.2.2 has the closed form solution

$$
\begin{aligned}
u(3n) &= \frac{3^{\left(\frac{1}{3}-2n\right)}}{3\Gamma(n+1)\Gamma\left(n+\frac{2}{3}\right)}, \\
u(3n+1) &= \frac{-3^{\left(\frac{2}{3}-2n\right)}}{9\Gamma(n+1)\Gamma\left(n+\frac{4}{3}\right)}, \\
u(3n+2) &= 0.
\end{aligned} \tag{AI.2.1.2.4}
$$

AI.2.2 Asymptotic expansion at ∞.

AI.2.2.1 First terms.

$$\mathrm{Ai}(x) \approx \mathrm{e}^{\left(\frac{-2}{3\xi^3}\right)}\sqrt{\xi}\left(\frac{1}{2\sqrt{\pi}} - \frac{5\xi^3}{96\sqrt{\pi}} + \dots\right)$$

where $\xi = \sqrt{\frac{1}{x}}$

AI.2.2.2 General form.

$$\mathrm{Ai}(x) \approx \mathrm{e}^{\left(\frac{-2}{3\xi^3}\right)}\sqrt{\xi}\sum_{n=0}^{\infty} u(n)\xi^n$$

where $\xi = \sqrt{\frac{1}{x}}$The coefficients $u(n)$ satisfy the following recurrence

$$16u(n)n + u(n-3)\left(-31 + 12n + 4(n-3)^2\right) = 0.$$

whose initial conditions are given by

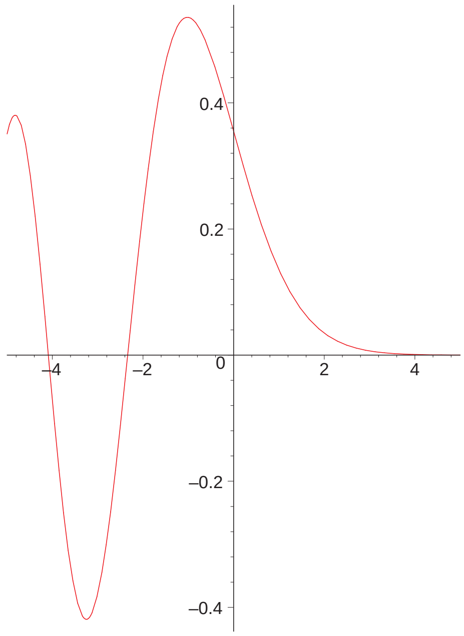
$$
\begin{aligned}
u(0) &= \frac{1}{2\sqrt{\pi}}, \\
u(1) &= 0, \\
u(2) &= 0.
\end{aligned}
$$

This recurrence has the closed form solution

$$
\begin{aligned}
u(3n) &= \frac{(-1)^n 6^{(2n)}\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{4\pi^{\frac{3}{2}}48^n\Gamma(n+1)}, \\
u(3n+1) &= 0, \\
u(3n+2) &= 0.
\end{aligned}
$$

AI.3 Graphs

AI.3.1 Real axis.



AI.3.2 Complex plane.