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The Degree Distribution of Bipartite Planar Maps and the Ising Model

Gilles Schaeffer
Lix, École Polytechnique (France)

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Summary by Julien Fayolle

Abstract

Enumerating bipartite (with black and white vertices) planar maps according to the degree distribution of the vertices is useful to physicists. We first exhibit a bijection between these maps and some family of trees. The generating functions of these trees are then obtained with classical decomposition on the combinatorial structure of the trees.

The physicists need to add *Ising* or *hard particle* models to planar maps to model particle location or spin. We can relate bijectively these maps with an additional structure to the bipartite maps. We finally enumerate the Ising and hard particle configurations on maps. (Joint work with Mireille Bousquet-Mélou from Labri)

1. Introduction

Following their bijective instincts the authors use in [1] a combinatorial approach to solve problems arising from 2-dimensional *quantum gravity*: the enumeration of planar maps under *Ising* and *hard particle* models. The main results have already been obtained by use of the powerful matrix integrals approach [3, 2]. Alas the matrix integral does not provide any insight on the combinatorial behavior of these maps like why their generating functions are algebraic. The authors exhibit a bijection between bipartite planar maps and some class of trees to explain the algebraicity of the generating functions. They then show the relation between bipartite planar maps and maps with the physicists' additional models.

Let's first recall some basic definitions: a *planar map* is a connected graph drawn on the sphere with non-intersecting edges, a *rooted map* is a map that possesses a root edge, i.e., an oriented edge. As a convention, when the map is drawn on the plane, its infinite face lies to the right of the rooted edge. In this talk we only deal with bipartite maps, meaning that each and every edge of the map has one black and one white endpoint.

The *degree distribution* of a bipartite map is a pair of partitions (λ, μ) coding the number of white vertices of a certain degree and the number of black vertices of a certain degree. The ith part of the partition λ is the number of white vertices of degree i. For example the map on Figure 2 has a contribution $x_2^2x_3x_4^2y_2^2y_4^2y_5$ to the generating function where x_i counts the number of white vertices of degree i and y_i the number of black vertices of degree i.

2. Maps and Trees

Direct enumeration of bipartite planar maps being difficult, we try a bijective argument: since planar maps are closely related to trees we enumerate a certain class of tree. Let's first introduce *blossom* trees: these are trees with black-and-white edges and also *half-edges*, which are called *leaves*

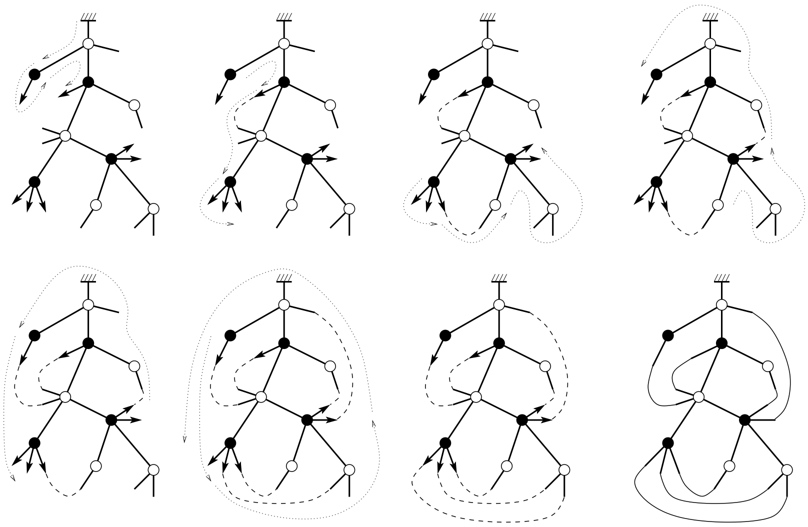

FIGURE 1. Closure

or *buds* whether they hang from a white or a black vertex. These trees are rooted at a leaf or a bud, and the vertex attached to this half-edge is called the root vertex. Furthermore, we define the *total charge* (resp. *charge*) of a blossom tree as the difference between the number of leaves in the tree and the number of buds where the root half-edge does (resp. does not) count. The charge of a vertex is just the charge of the subtree rooted at this vertex. A blossom tree must furthermore satisfy the condition that all white vertices have nonnegative (≥ 0) charge and all black ones a charge at most 1 (except the root vertex which has no charge condition to satisfy). A *k-leg map* is a bipartite map rooted at a half-edge with k leaves and no bud (hence the map is rooted at a leaf).

2.1. **Closure.** We define a closure operation ϕ on blossom trees T of total charge $k \geq 1$ by walking counterclockwise around the infinite face. Each and every time the next half-edge after a bud is a leaf we fuse these two half-edges into an edge. This process eventually stops and we obtain a bipartite planar map rooted at a half-edge with k leaves that remain unmatched, i.e., a k-leg map. Last but not least we introduce *balanced* trees: these trees are rooted at a leaf and after closure, the root is unmatched.

Proposition 1. *Let T be a balanced tree having total charge k ($k \geq 1$) and degree distribution (λ, μ). Then $\phi(T)$ is a k-leg map with degree distribution (λ, μ).*

Half-edges are counted as regular edges when determing the degree of a vertex. This is highly useful here since it keeps the degree distribution of the vertices unchanged by fusing half-edges (and also by splitting a bicolor edge as we will see in the next paragraph).

2.2. **Opening.** We shall now define the reciprocal operation of the mapping ϕ: the opening ψ of a k-leg map. Let M be a k-leg map, we walk counterclockwise around the infinite face starting from the root. For each edge visited from its black to its white endpoint that can be deleted without breaking the connectedness of the map, we cut this edge into two half-edges: a bud from the black endpoint and a leaf from the white one. At the end of the process we get a balanced tree (since it's rooted at a leaf (root of the k-leg map) and its root is a single leaf) of *total charge* (for this kind of charge we also count the rooted leaf) k.

Theorem 1. *Closure and opening are inverse bijection between balanced blossom trees of total charge k and k-leg maps. Moreover the degree distribution is preserved.*

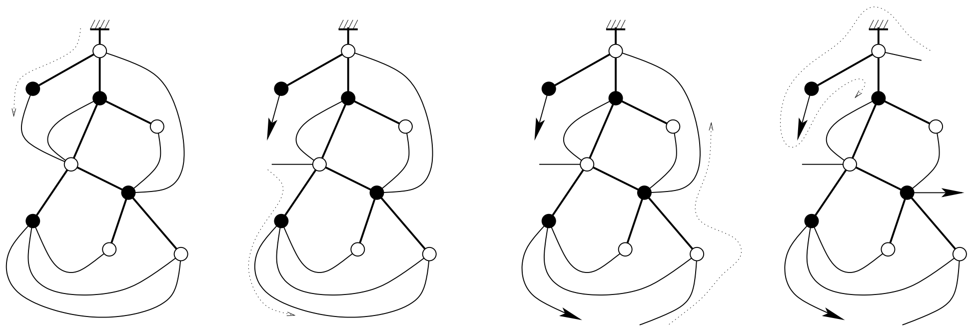

FIGURE 2. Opening

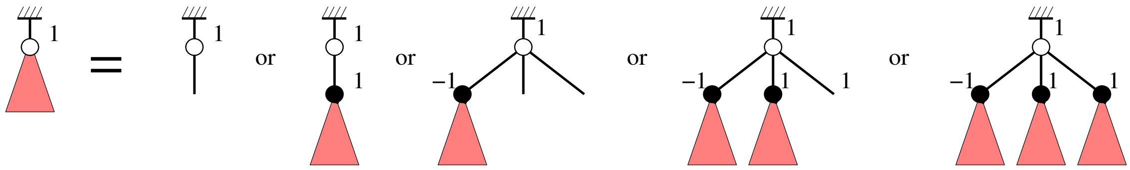

FIGURE 3. Decomposition

The reader is referred to [1] for a complete proof. We know that the generating functions of trees are algebraic, and this holds for balanced blossom trees of charge k. The bijection between blossom trees of total charge k and bipartite k-leg maps assures us that the generating function of k-leg maps is also algebraic.

3. Counting some Blossom Trees

We enumerate blossom trees according to the charge and the color of their root vertex. The methods are overall the same and we focus here on the degree generating function $A_1^{\circ-}(\mathbf{x},\mathbf{y})$ of blossom trees of charge 1 rooted at a white vertex (hence the "$\circ-$" exponent) where furthermore the vertices can have degree 2 or 4. We have a decomposition of these trees involving other blossom trees with degrees 2 or 4 as can be seen on Figure 3.

If the root vertex has degree 2 then one edge is the root leaf and the other one is either a leaf (charge 1) or a black and white edge (remember the tree is bicolor) with the charge of the tree hanging from the black endpoint being 1. If the root vertex has degree four, things follow the same pattern. We obtain a system with four degree generating functions:

$$A_1^{\circ-} = x_2(1 + A_1^{\bullet-}) + 3x_4A_{-1}^{\bullet-}(1 + A_1^{\bullet-})^2, \qquad A_3^{\circ-} = x_4(1 + A_1^{\bullet-})^3,$$
$$A_1^{\bullet-} = y_2A_1^{\circ-} + 3y_4(A_3^{\circ-} + (A_1^{\circ-})^2), \qquad A_{-1}^{\bullet-} = y_2 + 3y_4A_1^{\circ-}.$$

We solve this system to get $A_1^{\circ-}(\mathbf{x},\mathbf{y})$.

4. Ising and Hard Particle Models

We add an Ising configuration to a graph $\mathcal{G}$ by coloring its vertices with two colors (black and white). An edge is said *frustrated* if it has end points of the two colors. The hard particle model is a particularization of the Ising model where we forbid edges with two black endpoints ($\bullet - \bullet$). This means that no two physical particle can be adjacent in the graph.

We look for a bijection between maps with these two additional physical models (on the edges) and the bipartite maps we already obtained results on. On bipartite maps both $\circ - \circ$ and $\bullet - \bullet$

edges are forbidden. We use a trick to go from maps with a hard particle configuration to bipartite maps: we transform the $\circ - \circ$ edges of the map with a hard particle configuration into $\circ - \spadesuit - \circ$ edges in the bipartite map. The spades vertices are counted as ordinary black edges in the bipartite map.

The generating function related to the physics of the hard particle model is

$$H(\mathbf{X}, \mathbf{Y}, u) = \sum_{\mathcal{G}} \mathbf{X}^{\circ(\mathcal{G})} \mathbf{Y}^{\bullet(\mathcal{G})} u^{\circ-\circ(\mathcal{G})},$$

where each exponent respectively counts the degree of white vertices, black vertices and the number of $\circ - \circ$ edges in the map. The $\circ - \circ$ edges are said *vacant* because there is no physical particle on any end of the edge.

We want to obtain the generating function $H(\mathbf{X}, \mathbf{Y}, u)$ of maps with a hard particle configuration when the map is rooted at a vacant edge. Therefore when we use the transformation trick the bipartite map must be rooted at a black vertex of degree 2. We know from the previous sections how to obtain the degree generating function $F(\mathbf{x}, \mathbf{y})$ of bipartite maps rooted at a black vertex of degree 2.

Theorem 2. *The degree generating function $F(\mathbf{x}, \mathbf{y})$ of bipartite planar maps rooted at a black vertex of degree 2 is related to generating functions of blossom trees:*

$$F(\mathbf{x}, \mathbf{y}) = y_2((A_0^{\circ-} - A_2^{\bullet-})^2 + A_1^{\circ-} - A_3^{\bullet-} - (A_2^{\bullet-})^2).$$

Once we have this degree generating function [note that the generating functions $A_i^{\circ-}$ and $A_i^{\bullet-}$ are not the same as in section 3 since we have no condition on the degree of the vertices here], we notice that the transformation trick does not modify the degree of the white vertices hence $X_k = x_k$. The black vertices of degree $k \neq 2$ keep the same degree and the same color (these vertices are of the $\bullet$ type, not of the $\spadesuit$ type) thus $Y_k = y_k$. The black vertices of degree 2 can be of the two types: either it is a $\spadesuit$ vertex and then reversing the trick will transform it into a vacant edge marked by the variable u, or it is a $\bullet$ vertex and the degree remains the same. The leading y_2 in $F(\mathbf{x}, \mathbf{y})$ stems from the root condition and the black vertex at the root is of the $\spadesuit$ kind so we substitute $y_2 = u$ here. For the rest we substitute $y_2 = u + Y_2$ in the blossom trees generating functions. Once we applied the four substitutions in $F(\mathbf{x}, \mathbf{y})$ we obtain the degree generating function $H(\mathbf{X}, \mathbf{Y}, u)$.

For maps with the Ising model, we use a generalization of the transformation trick for the hard particle model, using two kind of white vertices. The substitution in the generating function counting bipartite maps is a little bit more complicated but the same method applies.

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