Profile of Random Recursive Trees and Random Binary Search Trees

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1. Introduction

Random recursive trees model is a simple probability model useful for many applications as system generation, spread of contamination of organisms, internet interface map, stochastic growth of networks or statistical physics. A random recursive tree is constructed as follows: one starts from a root node with the label 1; at each step \( n, n \geq 2 \), a new node with the label \( n \) is attached uniformly at random to one of the previous nodes labelled by \( 1, 2, \ldots, n-1 \). In this model, the labels are increasing along any path from the root to a node.

Denote by \( X_{n,k} \) the number of nodes at distance \( k \) from the root in a random recursive tree with \( n \) nodes. We are interested in the profile of random recursive trees, i.e., the collection \( \{ X_{n,k} : k \in \mathbb{N} \} \). More precisely, there exist some results about the mean and the variance of \( X_{n,k} \) but the limit distribution of \( X_{n,k} \) scaled by its mean, in the range \( k/\log n \sim \text{Constant} \) is a source of intriguing phenomena. Notice that the profile provides a fine and informative shape characteristic, it is related to path length, depth, height, width, \ldots and also to generation of random trees or other algorithmic problems.

A well-known connection between random recursive trees and binary search trees is the following: by rotation, a planar tree gives a binary tree and then it appears that the profile of random recursive trees is exactly the “left” profile of binary search trees (meaning that the distance of a node is only counted for left branches). Consequently, it is of the same flavour to study the profile of binary search trees; results and phenomena for \( Y_{n,k} \), the number of external nodes at level \( k \) and \( Z_{n,k} \), the number of internal nodes at level \( k \) in a random binary search tree, appear as a simple transposition.

2. Main Results

Let \( k \) depend on \( n \), let \( \alpha_{n,k} = k/\log n \) and suppose that:

\[
\lim_{n \to \infty} \alpha_{n,k} = \alpha
\]

The main phenomena are summarized in the following proposition:

**Proposition 1** (Main phenomena).

- \( \mathbb{E}(X_{n,k}) \) is unimodal and \( \mathbb{V}(X_{n,k}) \) is bimodal,
- \( \forall \alpha \in [0,e), X_{n,k}/\mathbb{E}(X_{n,k}) \xrightarrow{d} X_\alpha \) (convergence in distribution)
- \( \forall \alpha \in [0,1], X_{n,k}/\mathbb{E}(X_{n,k}) \xrightarrow{m} X_\alpha \) (convergence of all moments)
- for \( k = o(\log n) \), (case \( \alpha = 0 \)), \( (X_{n,k} - \mathbb{E}(X_{n,k}))/\sqrt{\mathbb{V}(X_{n,k})} \xrightarrow{m} \mathcal{N}(0,1) \)
for $k = \log n + o(\log n)$ (case $\alpha = 1$) and $|k - \log n| \to \infty$,

$$(X_{n,k} - E(X_{n,k}))/\sqrt{\text{Var}(X_{n,k})} \to X_1'$$

for $k = \log n + O(1)$, $(X_{n,k} - E(X_{n,k}))/\sqrt{\text{Var}(X_{n,k})}$ does not converge in distribution.

where $X_\alpha$ and $X_1'$ are limit distributions we describe further.

The proof is based both on the contraction method and the moment method.

Let $\mu_{n,k}$ be the first moment of $X_{n,k}$. A fine study of the asymptotics of $\mu_{n,k}$ leads to

$$\text{for } 0 \leq \alpha < e, \quad \frac{\log \mu_{n,k}}{\log n} \to \alpha - \alpha \log \alpha.$$  

The second moment and the variance of $X_{n,k}$ can also be asymptotically described: if $0 \leq \alpha < 2$,

$$\text{Var}(X_{n,k}) \sim \left(\frac{\Gamma(\alpha + 1)^2}{(1 - \alpha/2)\Gamma(2\alpha + 1)} - 1\right)\mu_{n,k}^2$$

and the variance exhibits a bimodal behavior when $\alpha = 1$.

For the second point in the proposition, i.e., the limit distribution, the starting point is the recurrence formula satisfied by the $X_{n,k}$ (a branching-type property):

$$X_{n,k} \overset{d}{=} X_{U_{n,k-1} + X_{n-U_{n,k}}}$$

where $U_n$ is uniform over $\{0,\ldots,n-1\}$ and $X_{j,k}$ and $X_{j',k'}$ are independent of each other and independent of $U_n$. As usually, thanks to the asymptotics of the first moment $\mu_{n,k}$ of $X_{n,k}$ and because $U_n/n$ converges in distribution to a uniform distribution $U$ on the interval $[0,1]$, it is possible to deduce a limit equation from (1):

$$X_{\alpha} \overset{d}{=} aU^\alpha X_{\alpha} + (1 - U)^\alpha X_{\alpha}.$$  

Moreover, the convergence of the first $m$ moments is obtained for $0 \leq \alpha < m/(m-1)$ and the moments of the limit distribution $\nu_m := E(X_{\alpha}^m)$ are given by a recurrence relation:

$$\nu_m = \frac{1}{m - \alpha^{m-1}} \sum_{j=1}^{m} \binom{m}{j} \nu_{j} \mu_{m-j} \alpha^{-1} \frac{\Gamma(j\alpha + 1)\Gamma((m - j)\alpha + 1)}{\Gamma(m\alpha + 1)}.$$  

In the particular case when $\alpha = 1$, the convergence of all moments in the fifth point of the Proposition is obtained by the method of moments: all moments satisfy the same type of recurrence:

$$a_{n,k} = b_{n,k} + \frac{1}{n-1} \sum_{j=1}^{n-1} (a_{j,k-1} + a_{j,k})$$

so that generating functions techniques and transfer theorem allow one to get asymptotics for higher moments. Notice that the limit distribution $X_1'$ is nothing but $(dX_{\alpha}/d\alpha)|_{\alpha=1}$ and is a solution of a ‘quicksort’-type equation:

$$X_1' \overset{d}{=} UX_1' + (1 - U)X_1'^* + U + U \log U + (1 - U) \log(1 - U).$$

Among open questions:

- what happens at the boundary of the interval $(\alpha_-, \alpha_+)$ of convergence of binary search trees (analog of the interval $(0, e)$ for recursive trees)?
- how to prove a.s. convergence in general?
- how to plot or simulate limit laws like $X_\alpha$?