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# Counting Unrooted Maps Using Tree Decomposition

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**Abstract**

This talk presents a new method to count unrooted maps on the sphere. This method is based on tree decomposition More precisely it gives enumerations of 2-connected and 3-connected unrooted maps with a complexity of $\mathcal{O}(N\log(N))$ for maps with $e \leq N$ edges and $\mathcal{O}(N^2)$ for maps with $i$ vertices and $j$ faces $i+j \leq N$. The family of 3-connected unrooted maps corresponds to the skeletons of polytopes in the 3D space, also called convex polyedra. This motivates us to find a good method to count these objects.

This work has been done with the help of Gilles Schaeffer.

## 1. Introduction

**Maps and roots.** A *map on the sphere* is the embedding of a graph on a sphere up to a continuous deformation. A *rooted map* is a map where one half-edge is marked.

Rooting greatly facilitates the enumeration by giving a starting point for a recursive decomposition. So, for example, we know for a long time the number of rooted planar maps:

$$\mathcal{M}'_n = \frac{2}{n+2}\frac{3^n(2n)!}{n!\,(n+1)!}$$

**Classical method for enumeration of unrooted objects.** Nevertheless some classical methods to count unrooted objects exist. One of them, introduced by Liskovets allows to obtain the number $c_n$ of unrooted maps with $n$ edges on the sphere by using Burnside's lemma (a result in group theory which is often useful in taking account of symmetry when counting mathematical objects) and the method of quotient. It gives $c_n$ as a function of $c'_n$, the number of rooted maps, and $c_n^{(k)}$, the number of $k$-rooted maps (A *$k$-rooted map* is a map having $k \geq 2$ undistinguishable roots.):

$$c_n = \frac{1}{2n}(c'_n + \sum_{k\geq 2}\phi(k)c_n^{(k)})$$

This formula adapts also for other families of maps (for example 2-connected and 3-connected maps). For the case of unconstraint maps on the sphere, $k$-rooted maps are easily counted by noticing that the quotient of a $k$-rooted map with respect to the symmetry is a rooted map. The number of preimages of a rooted map for this surjection is easy to calculate. Hence we obtain from this method, called quotient method and due to Liskovets, the enumeration of $k$-rooted maps.

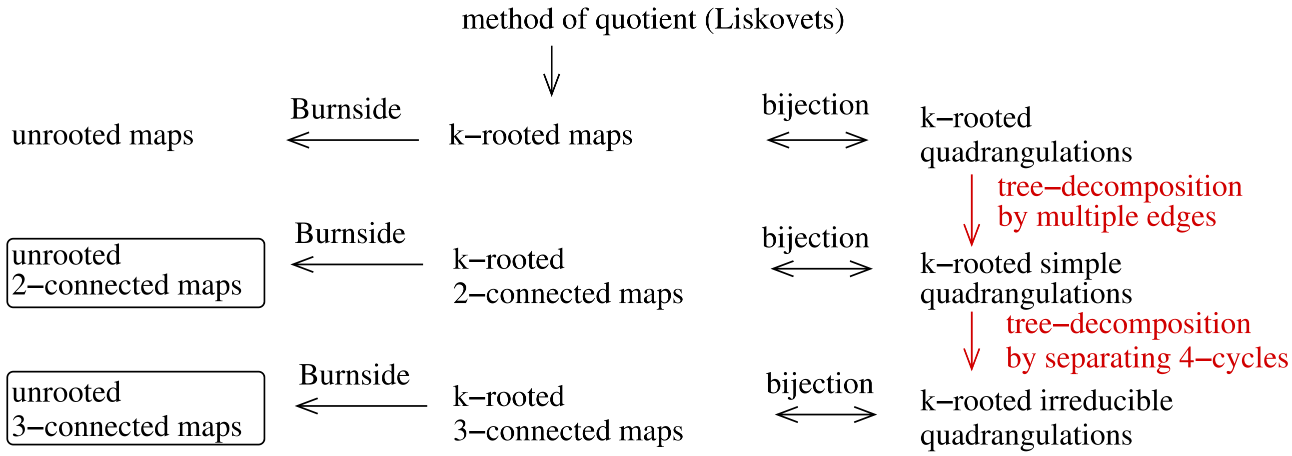


FIGURE 1. Scheme of the method.

**Types of $k$-rooted maps.** It was noticed by Liskovets that a $k$-rooted map has an embedding on the sphere wich is invariant by a certain rotation of angle $2\pi/k$ of the sphere. In addition, the two *poles* of the sphere crossed by the rotation-axis are either a vertex or the center of a face, and it can also be the middle of an edge if $k = 2$. These two points are called the *poles* of the $k$-rooted map. The *type* of the $k$-rooted map is the type of its two poles. For example, if the two poles are a vertex and a face, then the $k$-rooted map is said to have type face-vertex. In the particular case of a $k$-rooted quadrangulation, its type is more restricted. More precisely, the type is vertex-vertex if $k > 2$ and can also be face-vertex and face-face if $k = 2$.

**Bijection between maps and quadrangulation.** An other classical result that will be adapted to our problem is a wll-known bijection between maps and quadrangulations, which restricts well on 2-connected and 3-connected maps as illustrated on Figure 1.

**Results and methods.** Here will be only presented the method to count 2-connected unrooted maps. A similar one for 3-connected maps has also be done by the author. Figure 1 shows a summary of the results and methods used by the author. This new method of enumeration has a better complexity than previous ones. We will expose this point in our last section.

## 2. Enumeration of 2-Connected Unrooted Maps

One want here to count 2-connected unrooted maps. As in the classic case, using Burnside lemma, the enumeration comes down to count $k$-rooted 2-connected maps. Then one introduced a variation of the classical bijection between maps and quadrangulations: this bijection sets $k$-rooted 2-connected maps in bijection with $k$-rooted simple quadrangulations. We then use a tree decomposition method that will be explosed in detail in the following. We have now to deal with a family that we know how to enumerate (thanks to the bijection with the $k$-rooted maps and the method of the quotient).

**Method to perform the tree-decomposition.** We will transform an unrooted quadrangulation (that may have multiple edges) in a tree with two kinds of nodes: one representing the multiple edges and the other representing simple quadrangulations which are quadrangulations without multiple edges.

For each multiple edge of multiplicity $d$, we can imagine that we “blow,” from the interior of the sphere, each of the $d$ sectors delimited by the multiple edge. We so obtain $d$ quadrangulations

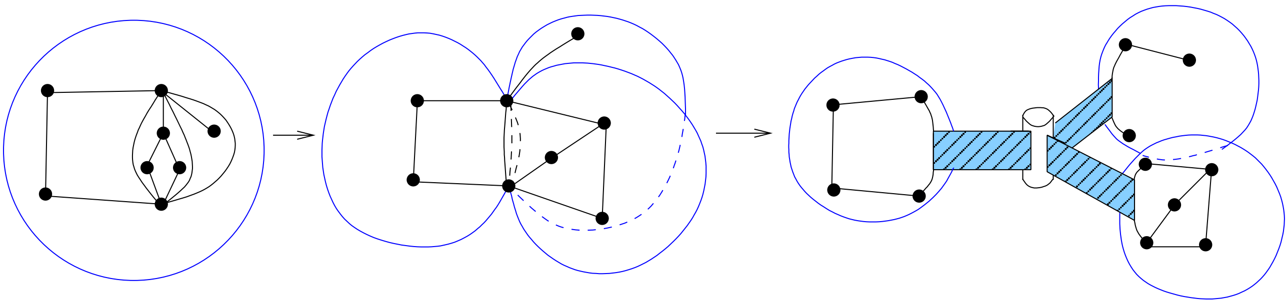

FIGURE 2. The tree-decomposition of a quadrangulation.

drawn on $d$ spheres that are now $d$ nodes of our tree. We connect them to a node representing the multiple edge. An example of this tree-decomposition can be seen in Figure 2. We then carry on the decomposition for each of the $d$ components.

**Equations.** With this method we obtain equations linking the generating functions (GF) of $k$-rooted quadrangulations (known) and the GF of $k$-rooted simple quadrangulations (unknown) of the same type. Here we present equations for quadrangulations of type vertex-vertex:

$$F_{vv}^{(k)}(z) = zf'(z)\frac{1}{1-f(z)} + \frac{(z(1+F(z)))'}{F(z)+1} g_{vv}^{(k)}\left(z(1+F(z)\right),$$

where $F_{vv}^{(k)}(z)$ is the GF of $k$-rooted quadrangulations of type vertex-vertex and $g_{vv}^{(k)}$ the GF of $k$-rooted simple quadrangulations of type vertex-vertex All the GF of the equation are known except $g_{vv}^{(k)}(z(1+F(z)))$. We deduct from that $g_{vv}^{(k)}(z(1+F(z)))$ and $g_{vv}^{(k)}(y)$ by doing the change of variables $y = z(1+F(z))$. Similarly, all GF of $k$-rooted simple quadrangulations can be obtained. Then, application of Burnside lemma (see Figure 1) allows to obtain the coefficients $c_n$ counting unrooted 2-connected maps:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_n$ | 2 | 2 | 3 | 6 | 16 | 42 | 151 | 596 | 2605 | 12098 | 59166 |

## 3. Algebraicity and Complexity

Here we prove that this enumeration of 2-connected and 3-connected maps can be done very quickly as enounced in Theorem 1. We first present here an efficient way to compute the GF of $k$-rooted simple quadrangulations. It is linear as opposed to the naive method in $\mathcal{O}(N^3)$. It is using the property of algebraicity of the GF of quadrangulations. Then we finish by showing that the global complexity of the theorem is implied by it.

**Theorem 1.** *In the case of the enumeration of 2-connected and 3-connected maps according to their number of edges, to obtain the first $N$ coefficients, we need $\mathcal{O}(N\log(N))$ operations.*

*In the case of the enumeration of 2-connected and 3-connected maps according to their number of faces and vertices, to obtain the table of the first coefficients with indexes $(i,j)$ with $i+j \leq N$, we need $\mathcal{O}(N^2)$ operations.*

**Efficient Expansion of the GF of $k$-rooted simple quadrangulations.** The starting point is the equation according to the number of faces:

$$F_{vv}^{(k)}(x) = xf'(x)\frac{1}{1-f(x)} + g_{vv}^{(k)}\left(x(1+F(x))^2\right)$$

As we know all the GF of the equation except $g_{vv}^{(k)}\left(x(1+F(x))^2\right)$, we can expand:

$$g_{vv}^{(k)}\left(x(1+F(x))^2\right) = 2x + 18x^2 + 180x^3 + \cdots .$$

We then do the change of variables: $y = x(1+F(x))^2$. We have $y = x + 4x^2 + 22x^3 + \cdots$ and $x(y) = y - 4y^2 - 10y^3 + \cdots$ and the equation becomes $g_{vv}^{(k)}(y) = 2y + 10y^2 + 56y^3 + \cdots$.

This naive method of computation has a complexity of $\mathcal{O}(n^3)$. We found a way to decrease it to linear. Instead of doing the change of variables directly between $x$ and $y$, we use small algebraic series $\beta(x)$ and $\eta(y)$ associated respectively to $x$ and $y$ such that the relation between $\beta$ and $\eta$ is rational.

To do so we will use the algebraicity of the GF. Let call $\beta$ the GF of blossoming trees. We have: $\beta(x) = x + 3\beta(x)^2$. The GF of rooted quadrangulations is a rational expression of $\beta(x)$:

$$F(x) = \frac{\beta(x)(2-9\beta(x))}{(1-3\beta(x))^2}.$$

All GF of $k$-rooted quadrangulations are also rational expressions of $\beta(x)$. So our starting equation becomes now:

$$g_{vv}^{(k)}\left(x(1+F(x))^2\right) = \frac{2\beta(x)}{1-6\beta(x)}.$$

The change of variables is $y = x(1+F)^2$, $y = \frac{\beta(1-4\beta)^2}{(1-3\beta)^3}$. We define $\eta$ as $\eta = \frac{\beta}{1-3\beta}$. We have $y = \eta(1-\eta)^2$. $\eta$ is an algebraic series in $y$ (serie of trees): $\eta(y) = \frac{y}{(1-\eta(y))^2}$. Furthermore $\beta = \frac{\eta}{1+3\eta}$. We substitute $\frac{\eta}{1+3\eta}$ to $\beta$ in $\frac{2\beta}{1-6\beta}$. We have at the end:

$$g_{vv}^{(k)}(y) = \frac{2\eta(y)}{1-3\eta(y)}.$$

From that we deduct a fast algorithm to compute the $N$ first coefficients of the serie $g_{vv}(y)$:

1. take the resultant of $\left\{\begin{array}{c} -\eta(1-\eta)^2 + y = 0 \\ -g_{vv}(1-3\eta) + 2\eta = 0 \end{array}\right.$ We have $4g_{vv}^3 + 8g_{vv}^2 - 8y - 36yg_{vv} - 54yg_{vv}^2 + 4g_{vv} - 27yg_{vv}^3 = 0$;
2. find a differential equation verified by $g_{vv}$ (We can use the function 'algeqtodiffeq' of 'gfun'): $g_{vv}(0) = 0, -4 - 6g_{vv}(y) + (2-54y)\frac{d}{dy}g_{vv}(y) + (-27y^2+4y)\frac{d^2}{dy^2}g_{vv}(y) = 0$;
3. take the coefficient $[y^n]$ in the equation to find a recursive equation for the coefficients (we can use the function 'diffeqtorec'): $(-6-27m-27m^2)u(m) + (6m+2+4m^2)u(m+1) = 0, u(0) = 0, u(1) = 2$.

**Global complexity of the coefficients computation.** The relation $2nc_n = c_n' + \sum_{k=2}^{n} \phi(k)c_n^{(k)}$ can be translated for the GF:

$$\sum_n 2nc_n y^n = g(y) + zg_{fv}(y^2) + z^2 g_{ff}(y^2) + \sum_{k=2}^{n} \phi(k) g_{vv}^{(k)}(y^k)$$

Let $\mathcal{C}_N(f)$ be the time of computation of the $N$ first coefficients of a function $f(z)$. We have

$$\mathcal{C}_N\left(\sum_n 2nc_n\right) = \mathcal{C}_N(g) + \mathcal{C}_{N/2}(g_{fv}) + \mathcal{C}_{N/2}(g_{ff}) + \sum_{k=1}^{n} \mathcal{C}_{N/k}\left(g_{vv}^{(k)}\right)$$

We have $\mathcal{C}_N = \mathcal{O}(N)$ for the GF of $k$-rooted simple quadrangulations. $\mathcal{C}_N\left(\sum_n 2nc_n y^n\right) = \mathcal{O}(N) + \mathcal{O}(N/2) + \sum_{k=1}^{N} \mathcal{O}(N/k)$ So we need $\mathcal{O}(N\log(N))$ operations.