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Particle Seas and Basic Hypergeometric Series

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Summary by Julien Fayolle

Abstract

The author introduces overpartitions and particle seas as a generalization of partitions. Both new tools are used in bijective proofs of basic hypergeometric identities like the q-binomial theorem, Jacobi[1]'s triple product, q-Gauß[2] equality or even Ramanujan[3]'s ${}_1\Psi_1$ summation.

1. Partitions

In 1969, G. E. Andrews was already looking for bijective proofs for some basic hypergeometric identities. The principle of bijective proofs is simple: if each side of an equation can be construed as a generating function counting some parameters for sets A and B of combinatorial objects, and we can go bijectively between objects of the two sets transforming the parameters on objects from A into the parameters of objects from B, then we have an identity.

Let us first recall the notation

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n) \quad \text{and} \quad (a;q)_k := \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{n=0}^{k-1}(1-aq^n).$$

If no ambiguity arises we note $(a;q)_k = (a)_k$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, we note $l(\lambda) = k$ its number of parts and $|\lambda| = \lambda_1 + \ldots + \lambda_k$ its weight. If nothing is specified, the partitions have positive parts (i.e., $\lambda_i > 0$). The generating functions for various types of partitions can be easily found in the literature (for example in [1])

The idea of introducing the particle seas stems from a bijective proof of Jacobi's triple product found by Itzykson (this proof was never published before [2]). We transform a usual partition into a *particle sea* (a precise and complete definition will be given later) by filling the zero-line with green balls and putting a blue ball in the column of ascissa x for any part x in the partition.

2. Overpartitions and the q-Binomial Theorem

In [3], Joichi and Stanton prove the q-binomial theorem through bijective means. We explain this bijection in terms of *overpartitions*, although the authors did not use this language.

Definition 1. An overpartition is a partition in which the *last* occurrence of a part *can* be overlined.

Remark. One can also define an overpartition with *first* instead of *last*.

[1]Karl Jacobi (1804–1851), German mathematician.
[2]Karl Friedrich Gauß (1777–1855), German mathematician, physicist, and astronomer.
[3]Srinivasa Aiyangar Ramanujan (1887–1920), Indian mathematician.

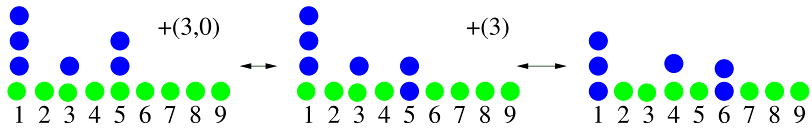

FIGURE 1. An example for Joichi and Stanton's bijection

For example, (9,9,$\overline{8}$,5,$\overline{5}$,3,3,1,1,$\overline{1}$) is an overpartition.

We first devise the generating function for the set O of overpartitions. We note that an overpartition can be decomposed bijectively in two parts: the overlined parts which make a partition into distinct parts (let D denote the set of partitions into distinct parts) and the non-overlined parts that are a general partition (denoted P). We introduce $o(\lambda)$ the number of overlined parts in the overpartition λ, and we have

$$\sum_{\lambda\in O} q^{|\lambda|}z^{l(\lambda)}x^{o(\lambda)} = \sum_{(\mu,\nu)\in D\times P} q^{|\mu|+|\nu|}z^{l(\mu)+l(\nu)}x^{l(\mu)} = \sum_{\mu\in D} q^{|\mu|}(zx)^{l(\mu)} \sum_{\nu\in P} q^{|\nu|}z^{l(\nu)} = (-qzx)_\infty \frac{1}{(zq)_\infty}.$$

For the reciprocal bijection one has to notice that the overlined parts are bound to be the last (or first, depending on the definition) ones with their respective weight.

We have to introduce additional notation: the set of overpartitions into k parts is $\mathcal{O}_k$, the set of partitions into distinct nonnegative parts smaller than k is $D_{k,\geq}$ and the set of partitions into parts less or equal to k is P_k.

Joichi and Stanton see an overpartition λ from $\mathcal{O}_k$ as the product of two partitions: one α from $P_k - P_{k-1}$ (i.e., the set of partitions into parts less or equal to k with at least one part of size k, this is in bijection with the set of partitions made of exactly k parts [it can be seen by rotating the partition diagram by $\pi/2$]), and one β from $D_{k,\geq}$. Here is the algorithm describing their bijection between $(P_k - P_{k-1}) \times D_{k,\geq}$ and $\mathcal{O}_k$:

- Transform α into a particle sea;
- For each part i in β, shift the first i blue balls in the sea by one to the right;
- Crash the $(i+1)$st ball on the zero-line (of the same column), destroying the green ball underneath it.

It follows that the partition λ has the characteristics $|\lambda| = |\alpha| + |\beta|$, $l(\lambda) = l(\alpha)$ and $o(\lambda) = l(\beta)$.

For the example in Figure 1 with $\alpha = (5,5,3,1,1,1)$, for $k = 6$ parts, and $\beta = (3,0)$. (We actually take a partition from the set of partitions in k parts but since this set is in bijection with $P_k - P_{k-1}$ we are OK.) We then have a sea particle that can be construed as an overpartition: For any blue ball on the zero-line, its abscissa is an overlined part in the overpartition, the other blue balls add a part in the overpartition equal to their abscissa. We obtain an overpartition into $k = 6$ positive parts $\lambda = (\overline{6}, 6, 4, \overline{1}, 1, 1)$.

This bijection means that

$$\sum_{\lambda\in\mathcal{O}_k} q^{|\lambda|}x^{o(\lambda)}z^{l(\lambda)} = \sum_{(\alpha,\beta)\in(P_k-P_{k-1})\times D_{k,\geq}} q^{|\alpha|+|\beta|}x^{l(\beta)}z^k = \sum_{\alpha\in P_k-P_{k-1}} q^{|\alpha|}z^k \sum_{\beta\in D_{k,\geq}} q^{|\beta|}x^{l(\beta)}$$

$$= z^k \left(\sum_{\alpha\in P_k} q^{|\alpha|} - \sum_{\alpha\in P_{k-1}} q^{|\alpha|} \right) (-x)_k = \frac{z^k q^k}{(q)_k}(-x)_k.$$

And from here we have

$$\sum_{\lambda \in O} q^{|\lambda|} x^{o(\lambda)} z^{l(\lambda)} = \sum_{k\geq 0} \sum_{\lambda \in \mathcal{O}_k} q^{|\lambda|} x^{o(\lambda)} z^{l(\lambda)} = \sum_{k\geq 0} \frac{z^k q^k (-x)_k}{(q)_k} = \frac{(-qzx)_\infty}{(zq)_\infty},$$

which is the q-binomial theorem.

Zeilberger also described a bijection to prove the q-binomial theorem that can be translated in the language of overpartitions.

3. Particle Seas

A particle sea can be interpreted as a couple of overpartitions with one overpartition into non-negative parts.

Definition 2. A particle sea is the upper half discrete plane partly filled with squares and balls such that:

- The zero-line is partially filled with squares and balls
- The rest of the positive quarter is partially filled with squares
- The rest of the non-positive quarter is partially filled with balls
- If the point (x, y) with $y > 0$ is filled, then the point $(x, y-1)$ is also filled
- If there is a ball (resp. square) of positive (resp. non-positive) abscissa then there is a square right (resp. ball left) to it.

Here the balls are graphically represented as green balls and the squares as blue balls.

Definition 3. The weight of a particle sea s is denoted $|s|$ and is the difference between the abscissæ of the squares of the positive quarter and the abscissæ of the balls of the non-positive quarter.

A *flat particle sea* is nothing but a sea particle with no ball or square above the zero-line. Itzykson's proof for Jacobi's triple product can be explained using only flat particle sea.

For a particle sea s, we define $d(s)$ (resp. $g(s)$) the number of squares (resp. balls) in the positive (resp. non-positive) quarter.

There exist a natural bijection between sea particles and pairs of overpartitions, one overpartition having positive parts: We split the sea particle into a positive and a non-positive side. For balls (resp. squares) in the non-positive (resp. positive) side we add their abscissa to their respective overpartition. If the ball (resp. square) in the non-positive (resp. positive) side is on the zero-line we overline its part.

4. q-Gauß

Let us use this new sea particle tool to prove the q-Gauß identity:

$$\sum_{n=0}^{\infty} \frac{(-1/b)_n (-1/a)_n (abq)^n}{(q)_n (q)_n} = \frac{(-aq)_\infty (-bq)_\infty}{(q)_\infty (abq)_\infty}. \tag{1}$$

Like for the q-binomial theorem, we identify each side of the identity as a product of generating functions of sets of overpartitions and sea particles. We are then left with providing a bijection between those sets. We note S_n instead of s for a sea particle s with $d(s) = g(s) = n$. The next algorithm allows us to go bijectively from the sea particle S_n to a pair of overpartitions (α, β) with additional conditions:

While there is a ball in the non-positive quarter do

$*$ Choose the lowermost ball in the leftmost column, its coordinates are (x, y)

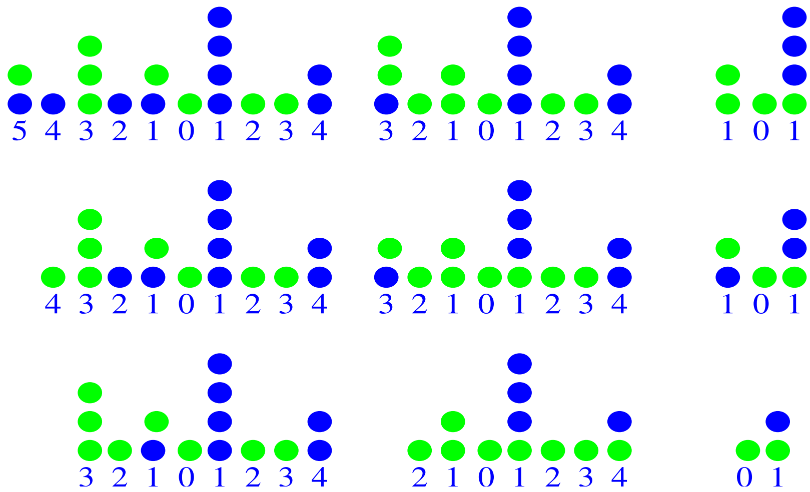

FIGURE 2. An example of the bijection for q-Gauß

- ∗ If there is a square on the zero-line of abscissa $x' = \min\{i, i > x\}$
  - · $i := x' - x$
  - · If $y = 0$, add a part i to α, change the square into a ball and the ball into a square
  - · else add a part $\bar{i}$ to α, destroy the ball, change the square into a ball
- ∗ Else find the uppermost square in the rightmost column, its coordinates are (x', y')
  - · $i := x' - x$
  - · If $y = 0$, add a part $\bar{i}$ to β, destroy the square, change the ball into a square
  - · else add a part i to β, destroy the ball and the square

The parameters of the overpartitions α and β are related to those on the original sea particle s by some conditions. The algorithm's steps are shown on picture 2 for some sea particle s and the final pair of overpartitions is $(\overline{7}, \overline{4}, 2, 2, \overline{1})$ and $(\overline{6}, \overline{2}, 2, \overline{1})$. It is easy to prove

$$\sum_{s \in S_n} q^{|s|} a^{g(s)-g_0(s)} b^{d(s)-d_0(s)} = \frac{(-1/b)_n(-1/a)_n}{(q)_n(q)_n}(abq)^n$$

and from there we use the bijection between sea particles from S_n and pairs of overpartitions with additional conditions to conclude.

5. Extensions

Sea particles have been used by the author to prove some more basic hypergeometric identities like Ramanujan's ${}_1\Psi_1$. The method is very similar the hard part being the search for the right bijection. The concept of particle sea has been extended to *colored particle sea* where, instead of having only balls and squares we can add some more colors to these. This leads to a more powerful tool and some harder results (related to ${}_6\phi_5$) have been obtained.

Bibliography

[1] Andrews (George E.). – *The Theory of Partitions.* – Addison–Wesley, 1976, *Encyclopedia of Mathematics and its Applications*, vol. 2.

[2] Corteel (Sylvie). – Particle seas and basic hypergeometric series. *Advances in Applied Mathematics*, vol. 31, 2003, pp. 199–214.

[3] Joichi (James T.) and Stanton (Dennis). – Bijective proofs of basic hypergeometric series identities. *Pacific Journal of Mathematics*, vol. 127, 1987, pp. 103–120.