Some Sharp Concentration Results about Random Planar Triangulations

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Submap Density Result

Let $T_0$ be any fixed triangulation, and $\eta_n(T_0)$ be the number of copies of $T_0$ in a random triangulation with $n$ vertices.

Richmond and Wormald (1988) : 

$$P(\eta_n(T_0) > cn) > 1 - e^{-\delta n}$$

for some positive constants $c$ and $\delta$. (depending on $T_0$)

Bender, Gao and Richmond (1992) : The above result holds for many families of maps.

Gao and Wormald : $\eta_n(T_0)$ is sharply concentrated around $cn$ for some constant $c$. 
**Theorem 1** Let $T_0$ be a 3-connected triangulation with $j + 3$ vertices such that there are $r$ distinct ways to root $T_0$. Let

$$c = 2r \left( \frac{27}{256} \right)^j.$$

Then

$$\mathbf{P} \left( |\eta_n(T_0) - cn| = o(cn) \right) \to 1,$$

provided that $cn \to \infty$. 
Define
\[ \mu_k = \frac{8(k - 2)}{4k^2 - 1} \left( -\frac{3}{4} \right)^k \left( -\frac{3/2}{k} \right). \]

**Theorem 2** Let \( M \) be a 3-connected near-triangulation with external face of degree \( k \) and with \( j \) internal vertices such that there are \( r \) distinct ways to root the external face. Then, for fixed \( j, k \) with \( k \geq 4 \),
\[
P \left( \left| \eta_n(M) - r\mu_k \left( \frac{27}{256} \right)^{j-1} n \right| = o(n) \right) \to 1.
\]
A sketch of the proof.

First study the number $\zeta_n(k)$ of vertices of degree $k$ in a random triangulation with $n + 2$ vertices.

- $T_n$ denotes the number of rooted triangulations with $n + 2$ vertices.

- $T_{n,k}$ denotes the number of rooted triangulations with $n + 2$ vertices and root vertex of degree $k$.

- $T_{n,k,l}$ denotes the number of rooted triangulations with $n + 2$ vertices, root vertex of degree $k$ and another distinguished vertex of degree $l$. 
Step 1 Use combinatorial argument to show

$$E(\zeta_n(k)) = \frac{6nT_{n,k}}{kT_n},$$

$$E(\zeta_n(k)(\zeta_n(k) - 1)) = \frac{6nT_{n,k,k}}{kT_n}.$$  

Step 2 Obtain functional equations for the generating functions for $T_{n,k,l}$, $T_{n,k}$ and $T_n$, and perform singularity analysis.
Step 3 Derive a multivariate version of Flajolet and Odlyzko’s transfer theorem, and obtain the following asymptotics:

\[
T_n = \frac{\sqrt{6}}{32\sqrt{\pi}} n^{-5/2} (256/27)^n (1 + O(1/n)),
\]

\[
T_{n,k} = \frac{k \sqrt{6}}{192 \sqrt{\pi}} \mu_k n^{-5/2} (256/27)^n \left( 1 + O \left( k^{20}/n \right) \right),
\]

\[
T_{n,k,k} = \frac{k \sqrt{6}}{192 \sqrt{\pi}} \mu_k^2 n^{-3/2} (256/27)^n \left( 1 + O \left( k^{20}/n \right) \right),
\]

uniformly for \( k = O(\log n) \).
Step 4 Derive asymptotics for the first two moments of $\zeta_n(k)$.

$$E(\zeta_n(k)) = n\mu_k \left(1 + O\left(k^{20}/n\right)\right),$$

$$V(\zeta_n(k)) = n\mu_k + (n\mu_k)^2 O\left(k^{20}/n\right),$$

uniformly for all $k = O(\log n)$.

It follows from Chebyshev’s inequality that

$$\mathbb{P}\left(|\zeta_n(k) - \mu_k n| = o(\mu_k n)\right) \to 1$$

uniformly for all

$$k < (\log n - (1/2) \log \log n)/\log(4/3) - \Omega(n).$$
Lemma 1  Let $T_0$ be a 3-connected triangulation with $j + 3$ vertices such that $j = o(n)$ and there are $r$ distinct ways to root $T_0$. Let $\eta_n(T_0)$ be the number of copies of $T_0$ in a random rooted triangulation with $n + 2$ vertices. Then

$$E(\eta_n(T_0)) = r \left( \frac{27}{256} \right)^{j-1} E(\zeta_{n+1-j}(3))(1 + o(1)),$$

$$E(\eta_n(T_0)(\eta_n(T_0) - 1)) = r^2 \left( \frac{27}{256} \right)^{2j-2} \times E(\zeta_{n+2-2j}(3)(\zeta_{n+2-2j}(3) - 1))(1 + o(1)).$$
Maximum Vertex Degree

Let $\Delta_n$ be the maximum vertex degree of a random map in a family of maps of size $n$.

Devroye, Flajolet, Hurtado, Noy and Steiger showed that, for triangulations of an $n$-gon,

$$\mathbb{P} \left( \left| \Delta_n - \log n / \log 2 \right| \leq (1 + \epsilon) \log \log n / \log 2 \right) \to 1$$
Gao and Wormald (to appear in JCT-A) showed

- for triangulations of an $n$-gon,
  \[
  P \left( |\Delta_n - (\log n + \log \log n)/\log 2| \leq \Omega(n) \right) \to 1
  \]

- for 3-connected triangulations of $n$ vertices,
  \[
  P \left( |\Delta_n - (\log n - (1/2) \log \log n)/\log(4/3)| \leq \Omega(n) \right) \to 1
  \]

- for all maps of $n$ edges,
  \[
  P \left( |\Delta_n - (\log n - (1/2) \log \log n)/\log(6/5)| \leq \Omega(n) \right) \to 1
  \]
Parallel Results about Lattice Walks

Neal Madras recently proved a very nice result about patterns in lattice clusters which is parallel to the submap density results. Let $C_n$ be a set of lattice clusters of size $n$, and let $P_0$ be any fixed pattern. Then there is a positive constant $\epsilon$ such that the fraction of clusters that contain less than $\epsilon n$ copies of $P_0$ (translations of $P_0$) is exponentially small.
Madras believes that the number of copies should be sharply concentrated around $cn$ for some positive constant $c$. (which he calls the law of large numbers)
**Other Sharp Concentration Results about Triangulations**

An example of diagonal flips and flippable edges. Let $\zeta_n$ be the number of flippable edges in a random 2-connected triangulation (3-connected triangulation) of $n$ vertices.

Gao and Wang (to appear in JCT-A): $\zeta_n$ is sharply concentrated around $5n/2$ ($9n/4$).
\( \epsilon \) will denote a small positive constant, \( \phi \) is a constant satisfying \( 0 < \phi < \pi/2 \), and \( y = (y_1, y_2, \ldots, y_d) \).

Define

\[
\Delta_x(\epsilon, \phi) = \{ x : |x| \leq 1 + \epsilon, \ x \neq 1, \ |\text{Arg}(x - 1)| \geq \phi \},
\]

\[
\Delta_j(\epsilon, \phi) = \{ y_j : |y_j| \leq 1 + \epsilon, \ y_j \neq 1, \ |\text{Arg}(y_j - 1)| \geq \phi \},
\]

\[
\mathcal{R}(\epsilon, \phi) = \{(x, y) : |y_j| < 1, 1 \leq j \leq d, x \in \Delta_x(\epsilon, \phi) \}.
\]

Let \( \beta_j > 0 \) for \( 1 \leq j \leq d \), and \( \alpha \) be any real number.
**Definition 1.** We write

\[ f(x, y) = \tilde{O} \left( (1 - x)^{-\alpha} \prod_{j=1}^{d} (1 - y_j)^{-\beta_j} \right) \]

if there are \( \epsilon > 0 \) and \( 0 < \phi < \pi/2 \) such that in \( \mathcal{R}(\epsilon, \phi) \)

(i) \( f(x, y) \) is analytic, and

\[ f(x, y) = O \left( |1 - x|^{-\alpha} \prod_{j=1}^{d} (1 - |y_j|)^{-\beta_j} \right) \]

as \( (1 - x)(1 - y_j)^{-p} \to 0 \), for \( 1 \leq j \leq d \), and some \( p \geq 0 \).

(ii) \[ f(x, y) = O \left( |1 - x|^{-\alpha'} \prod_{j=1}^{d} (1 - |y_j|)^{-q} \right) \]

for some \( q \geq 0 \) and some real number \( \alpha' \).
Definition 2. We write
\[ f(x, y) \approx c (1 - x)^{-\alpha} \prod_{j=1}^{d} (1 - y_j)^{-\beta_j} \]
if \( f(x, y) \) can be expressed as
\[
\begin{align*}
f(x, y) &= c(y)(1 - x)^{-\alpha} \prod_{j=1}^{d} (1 - y_j)^{-\beta_j} \\
&\quad + \sum_{j=0}^{d} C_j(x, y) + E(x, y)
\end{align*}
\]
such that

(i) \( C_0(x, y) \) is a polynomial in \( x \), and for \( 1 \leq j \leq d \), \( C_j(x, y) \) is a polynomial in \( y_j \).

(ii) \[
E(x, y) = \tilde{O}\left( (1 - x)^{-\alpha'} \prod_{j=1}^{d} (1 - y_j)^{-\beta'_j} \right),
\]
for some \( \alpha' < \alpha \) and \( \beta'_j \geq 0, 1 \leq j \leq n \).
(iii) $c(y) = c + O\left(\sum_{j=1}^{d} |1 - y_j|\right)$ and is analytic in $\{y : y_j \in \Delta_j(\epsilon, \phi)\}$, and $c(1) = c \neq 0$. 
Lemma 2 Suppose

\[ f(x, y) = \tilde{O}\left((1 - x)^{-\alpha} \prod_{j=1}^{d} (1 - y_j)^{-\beta_j}\right). \]

Then

(i) as \( n \to \infty \) and \( 1 \leq k_j = O(\log n) \) (\( j = 1, \ldots, d \)),

\[ \left[ x^n y^k \right] f(x, y) = O\left(n^{\alpha - 1} \prod_{j=1}^{d} k_j^\beta_j\right); \]

(ii) for any \( 0 < \epsilon' < 1 \) and all \( n, k_j \),

\[ \left[ x^n y^k \right] f(x, y) = O\left(n^{\alpha - 1} \prod_{j=1}^{d} (1 - \epsilon')^{-k_j}\right). \]
Lemma 3 Let $d \geq 1$ and

$$f(x, y) \approx c \ (1 - x)^{-\alpha} \prod_{j=1}^{d} (1 - y_j)^{-\beta_j},$$

where $\alpha$ is neither a negative integer nor 0, and $c \neq 0$. Then as $n \to \infty$ and $k_j = O(\log n)$ ($j = 1, \ldots, d$),

$$[x^n y^k] f(x, y) = \frac{c}{\Gamma(\alpha)} \prod_{j=1}^{d} \left( k_j^{-1} \beta_j / \Gamma(\beta_j) \right) n^{\alpha-1}$$

$$\times \left( 1 + O \left( \sum_{j=1}^{d} (1/k_j) \right) \right).$$