Enumeration of Planar Triangulations

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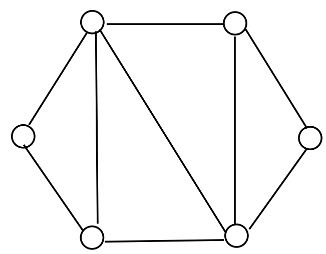
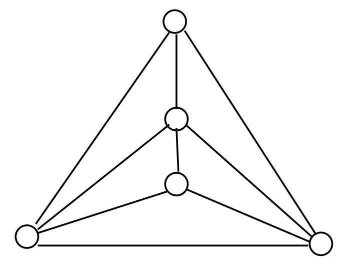
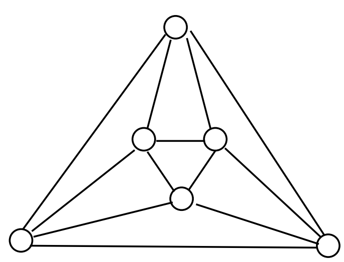
Definition and Examples



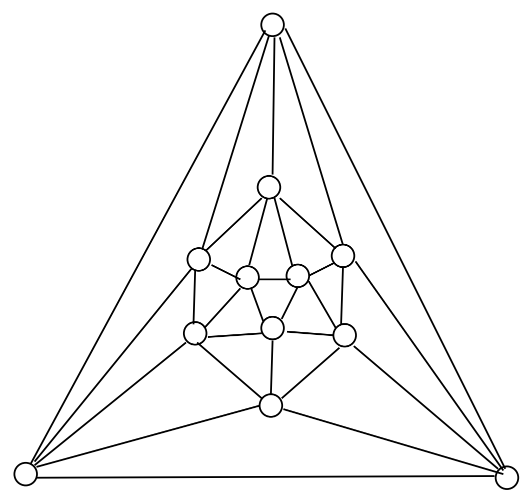
Triangulation of a polygon



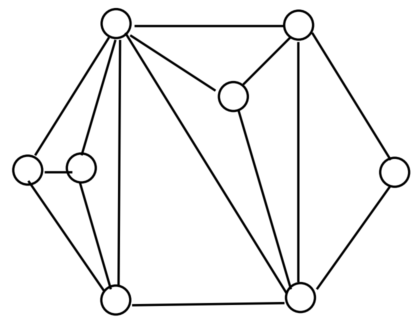
A triangulation



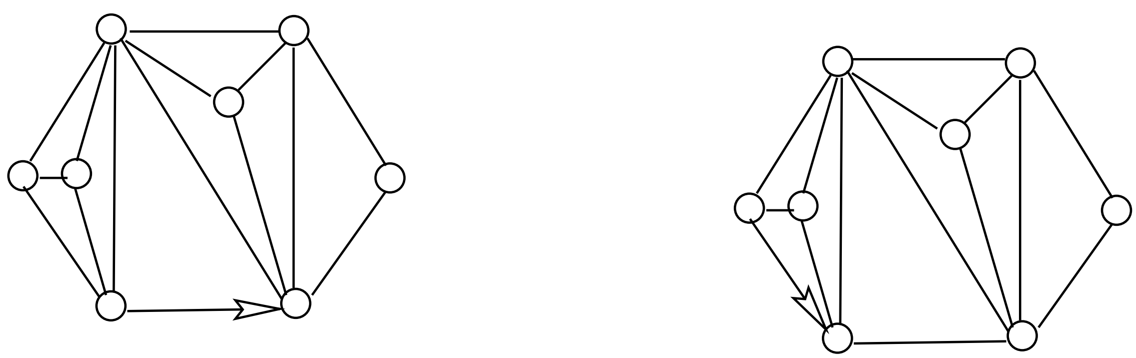
The octahedron



The icosahedron



Near-triangulation



Rooted near-triangulations

Background

- Euler (1870) enumerated triangulations of polygons :

$$\frac{1}{n-1}\binom{2(n-2)}{n-2}$$

- Tutte (1960s) enumerated 3-connected triangulations (among other families of maps) :

$$\frac{1}{(2n+1)(4n+3)}\binom{4n+3}{n+1}$$

  Motivated by the 4-colour problem.

- (1980s) Enumeration of maps on general surfaces. (Largely motivated by works in quantum physics)

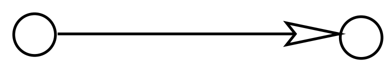
Maps have rich structures which give rise to many interesting and difficult problems whose solutions appeal to techniques from

- topological graph theory,
- Combinatorial enumeration,
- Algebra,
- Asymptotic analysis,
- Probability theory,
- Special functions.

Recursive approach.
Example. Enumeration of all planar triangulations.

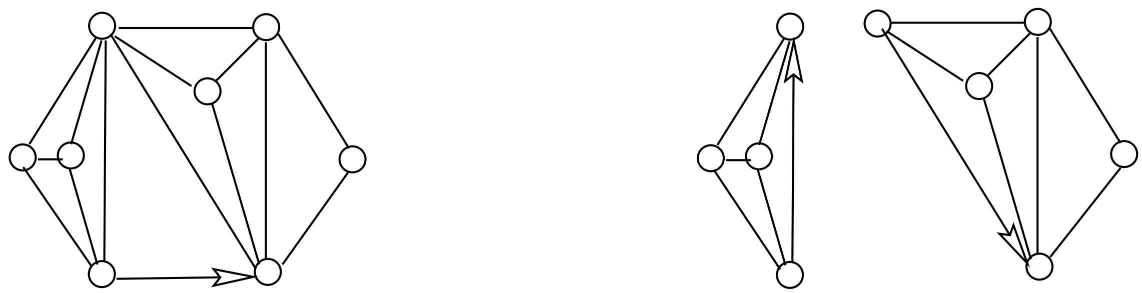
Let $F_{m,n}$ be the number of rooted planar near-triangulations with m exterior vertices and n interior vertices. Define the generating function

$$F(x,y) = \sum_{m,n} F_{m,n} x^n y^n = \sum_m F_m(x) y^m$$



The degenerate triangulation

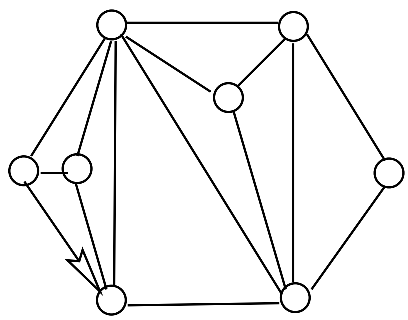
Removing the root edge gives two smaller near-triangulations.

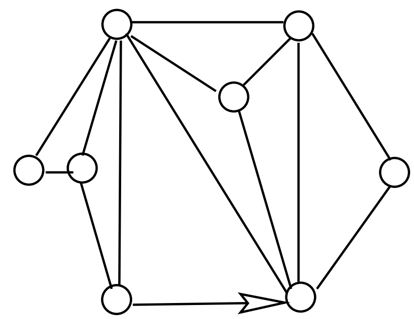


The contribution from this case is

$$y^{-1}F^2(x, y)$$

Removing the root edge gives one smaller near-triangulation.





The contribution from this case is

$$xy^{-1}\left(F(x,y) - y^2 - yF_3(x)F(x,y)\right)$$

Quadratic method.

$$\begin{aligned} F(x,y) &= y^2 + y^{-1}F^2(x,y) \\ &\quad + xy^{-1}\left(F(x,y) - y^2 - yF_3(x)F(x,y)\right) \end{aligned}$$

Completing the square for $F(x,y)$, we rewrite the above equation as $A^2 = B$, where

$$\begin{aligned} A &= F + \frac{x(1-yF_3)-y}{2}, \\ B &= \left(\frac{x(1-yF_3)-y}{2}\right)^2 + xy^2 - y^3. \end{aligned}$$

Find a power series $y = y(x)$ such that

$$A(x, y(x)) = 0.$$

Then

$$B(x, y(x)) = B_y(x, y(x)) = 0.$$

Solving the system, we obtain

$$\begin{aligned} y(x) &= t(1-t)^2, \\ F_3(x) &= (1-2t)(1-t)^{-3}, \end{aligned}$$

where $t = t(x)$ is a power series in x satisfying

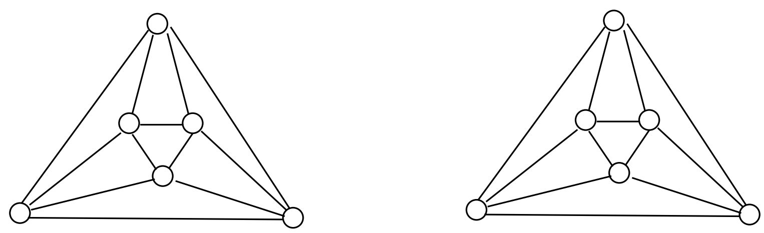
$$x = t(1-t)^3.$$

Apply Lagrange inversion formula to obtain Tutte's result.

Compositional approach.

Example. 4-connected planar triangulations.

Removing the vertices inside each maximal separating triangle, one obtains 4-connected triangulations (excluding K_3). Conversely, replacing each face of a 4-connected triangulation with a 3-connected triangulation, one gets 3-connected triangulations (excluding K_3).



If we use $G(x)$ to denote the generating function for 4-connected triangulations, where x marks the number of interior vertices. Then

$$F_3(x) - 1 = \sum_{n\geq 1} G_n x^n F_3^{2n+1} = F_3 G\left(xF_3^2\right).$$

Using the parametric expression for $F_3(x)$, we obtain

$$G(z) = 1 - (1-t)^3(1-2t)^{-1},$$

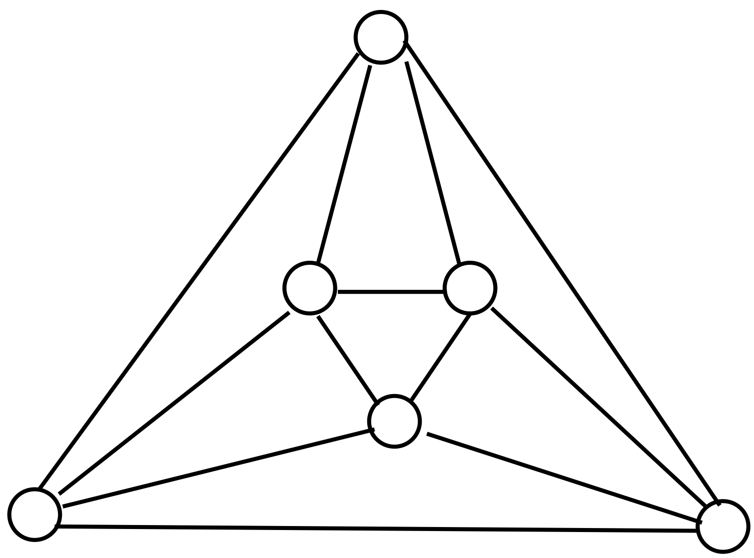
where

$$z = t(1-2t)^2(1-t)^{-3}.$$

From 4-connected triangulations to 5-connected triangulations ?

Two major problems.

- Removing interior vertices from separating 4-cycles, we get faces of degree 3 or 4.
- Two maximal separating 4-cycles may overlap.



Definition. A map is of $\{3,4\}$-type if it contains no loops or multiple edges and has all interior faces of degree 3 or 4. Let $M(x, y, z)$ be the generating function for $\{3,4\}$-type maps, where x and y marks the number of interior faces of degree 3 and 4, respectively, and z marks the root face degree. Note

$$M(x, y, z) = 1 + z^2 + M_3(x, y)z^3 + M_4(x, y)z^4 + O(z^5)$$

The recursive decomposition technique gives

$$\begin{aligned} M &= 1 + z^2M^2 + xz^{-1}(M - 1 - z^2M - zM_3(M-1)) \\ &\quad + yz^{-2}(M - 1 - z^2 - z^3M_3M - z^2M_4(M-1)). \end{aligned}$$

Rewrite the above equation as $A^2 = B$, where

$$\begin{aligned} A &= 2z^4M - (xz^2 + yz^3)M_3 - yz^2M_4 + xz + y - xz^3 - z^2, \\ B &= (-4yM_4 - 4xM_3 + 2xyM_3 + x^2 - 4 + y^2M_3^2 + 4y)z^6 \\ &\quad + (6x + 2x^2M_3 + 2yM_3 + 2xyM_4 + 2y^2M_3M_4 + 2xyM_3^2)z^5 \\ &\quad + (2yM_4 + x^2M_3^2 + y^2M_4^2 + 2xM_3 - 2xyM_3 \\ &\qquad + 1 + 2xyM_3M_4 - 2x^2 + 4y)z^4 \\ &\quad + (-2xy - 2y^2M_3 - 2xyM_4 - 2x^2M_3 - 2x)z^3 \\ &\quad + (x^2 - 2y - 2y^2M_4 - 2xyM_3)z^2 + 2xyz + y^2. \end{aligned}$$

Note that the standard quadratic method does not work here, because there are two unknown functions M_3 and M_4. By Brown's theorem, there are power series

$$R(x,y,z) = 1 - 4\sum_{j=1}^{r} R_j(x,y)z^j$$

and

$$Q(x,y,z) = \sum_{j=0}^{q} Q_j(x,y)z^j$$

such that

$$B = RQ^2.$$

Here it is natural to try $r = q = 2$.

Now we look for

$$R(x,y,z) = 1 - 4R_1(x,y)z - 4R_2(x,y)z^2,$$

and

$$Q(x,y,z) = Q_0(x,y) + Q_1(x,y)z + Q_2(x,y)z^2,$$

such that $A^2 = B = RQ^2$. That is

$$AR^{-1/2} = Q.$$

Expanding the left hand side as a power series in z, noting

$$A = 2z^4(1+z^2) - (xz^2+yz^3)M_3 - yz^2M_4 + xz + y - xz^3 - z^2 + O(z^7),$$

and comparing the coefficients of z^j, $3 \le j \le 6$, we obtain four equations with four unknown functions M_3, M_4, R_1, R_2.

Using Computer Algebra System such as **Maple**, we can obtain

$$\begin{aligned} R_1 &= \frac{u}{2} \\ R_2 &= 1 + v - \frac{u^2}{4}, \\ M_3 &= u^3 - 2uv + u, \\ M_4 &= 3u^4 - 5u^2v + u^2 - v^2 + v + 2. \end{aligned}$$

where

$$\begin{aligned} x &= \frac{3u^3 - 2uv + u}{(1+v)^3}, \\ y &= \frac{v - u^2}{(1+v)^3}. \end{aligned}$$

Deal with separating triangles

Let $\tilde{M}_j(\tilde{x}, \tilde{y})$ denote the generating function for {3,4}-type maps with no separating triangles and with root face degree j, where $\tilde{M}_3(\tilde{x}, \tilde{y})$ excludes the single triangle map, and $\tilde{M}_4(\tilde{x}, \tilde{y})$ excludes the 2-path map.

The compositional approach gives

$$\begin{aligned} M_3(x,y) - x &= \tilde{M}_3(\tilde{x}, \tilde{y}), \\ M_4(x,y) - 2 &= \tilde{M}_4(\tilde{x}, \tilde{y}), \end{aligned}$$

where

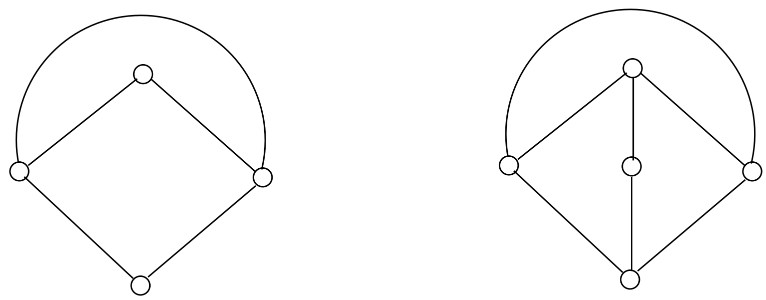
$$\tilde{x} = M_3(x,y), \quad \tilde{y} = y.$$

Using the parametric expressions for $M_3(x,y)$ and $M_4(x,y)$, we obtain

$$\begin{aligned} \tilde{x} &= u^3 - 2uv + u, \\ \tilde{y} &= \frac{v - u^2}{(1+v)^3}, \\ \tilde{M}_3(\tilde{x}, \tilde{y}) &= u^3 - 2uv + u - \frac{3u^3 - 2uv + u}{(1+v)^3}, \\ \tilde{M}_4(\tilde{x}, \tilde{y}) &= 3u^4 - 5u^2v + u^2 - v^2 + v. \end{aligned}$$

Deal with separating quadrangles

Let ${M^*}_3(x^*, y^*)$ denote the generating function for {3,4}-type maps with no separating triangles or separating quadrangles, and with root face degree 3, with the following cases being excluded.



The compositional approach gives the following:

$$\begin{aligned} {M^*}_3(x^*, y^*) &= \tilde{M}_3(\tilde{x}, \tilde{y}) - 3\tilde{x}\tilde{y} \\ &\quad - \left(\frac{3\tilde{x}(y^* + \tilde{x}^2)^2}{1 + y^* + \tilde{x}^2} - 3\tilde{x}^5 - 3\tilde{x}^3 y^* + \tilde{x}^7 \right), \end{aligned}$$

where

$$x^* = \tilde{x}, \quad y^* = \tilde{M}_4(\tilde{x}, \tilde{y}) - 2\tilde{x}^2.$$

Setting $y^* = 0$ and defining $T(w) = x^* {M^*}_3(x^*, 0)$, with $w = {x^*}^2$, we obtain

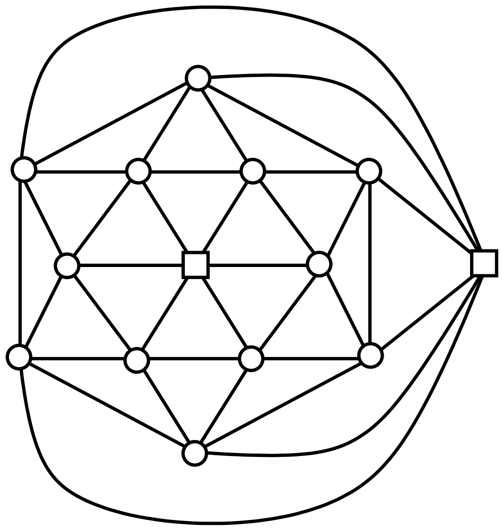
$$\begin{aligned} T(w) \;=\;& 4w(s+1)^3 \frac{w(3s-1)+(3s-2)(s+1)^3}{(w-s^3+3s+2)^3} \\ & -\frac{3w^3}{1+w} + 3w^3 - w^4 + w, \end{aligned}$$

where

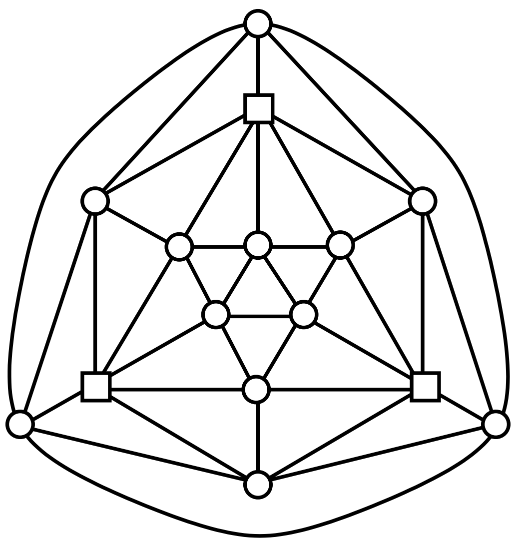
$$w^2 - 2(4s^2+2s+1)(s+1)^2 w - s(s+2)(s+1)^4 = 0.$$

$$
\begin{aligned}
T(w,0) \;=\; & w^2 + w^{10} + 6w^{12} + 13w^{13} + 55w^{14} + 189w^{15} \\
& +694w^{16} + 2516w^{17} + 9213w^{18} \\
& +33782w^{19} + 124300w^{20} + 458502w^{21} \\
& +1695469w^{22} + 6284175w^{23} + 23344173w^{24} \\
& +86904615w^{25} + 324197100w^{26} + 1211841846w^{27} \\
& +4538611107w^{28} + 17029834923w^{29} \\
& +64014608376w^{30} + 241046175666w^{31} \\
& +909171583214w^{32} + 3434698413540w^{33} \\
& +12995770332449w^{34} + 49244814205978w^{35} \\
& +186869902642338w^{36} + 710092414631410w^{37} \\
& +2701869001901348w^{38} + 10293565023621642w^{39} \\
& +39264367207303736w^{40} + 149948715603475020w^{41} \\
& +573296447054198853w^{42} + 2194269537140318814w^{43} \\
& +8407321079482344885w^{44} \\
& +32245250079373145325w^{45} \\
& +123793719740320431351w^{46} \\
& +475708871388447892665w^{47} \\
& +1829697901763800534137w^{48} \\
& +7043676860459459010231w^{49} \\
& +27138658500426617820826w^{50} + \cdots
\end{aligned}
$$

(a)



(b)



Asymptotics.

If we use $T_j(n)$ denote the number of rooted planar j-connected triangulations with n vertices. Then

$$T_j(n) = c_j n^{-5/2} r_j^{-n} \left(1 + O(1/n)\right),$$

where c_j is a positive constant,

$$r_3 = 27/256, \quad r_4 = 4/27,$$

and

$$r_5 = \frac{-(s+1)^2(3s^2+6s+1)}{2(8s^2+7s+2)} \approx 0.24775354,$$

with $s \approx -0.32185122$ satisfying

$$128s^6 + 336s^5 + 333s^4 + 304s^3 + 210s^2 + 72s + 9 = 0.$$

Note that the above equation is not solvable.