Continued Fractions, Comparison Algorithms and Fine Structure Constants

Philine Flajolet
Algorithms Proj ct, INRIA Rocquencourt
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Summary by Cyril Banderier

Abstract
The simple problems of comparing fractions (Gosper’s algorithms for continued fractions from the Hacker’s Memorandum) and of deciding the orientation of triangles in computational geometry lead to a complexity analysis with an excursion into a surprising variety of domains: dynamical systems (symbolic dynamics), number theory (continued fractions), special functions (multiple zeta values), functional analysis (transfer operators), numerical analysis (series acceleration), and complex analysis (Riemann hypothesis). These domains all eventually contribute to a detailed characterization of the complexity of comparison and sorting algorithms, either on average or in probability. (Joint work with Brigitte Vallée.)

1. Introduction
To compare two rational numbers (or similarly, to determine the sign of a $2 \times 2$ determinant), a rational problem in computational geometry is a decision problem when you have to work with a numerical calculator limit $d$ to a given number $r$ of digits. For example, since $\frac{31680}{9063} = \frac{83719}{65351} \approx 3 \times 10^{-11}$, a computation with 10-digit accuracy cannot compare “naively” the two rational numbers.

In the “Hackers’ Morandum” [2], it is shown that it is always possible to solve this comparison problem without changing the accuracy of the calculator. The algorithm consists in expanding both rational numbers in continued fractions, but stopping as soon as one of them is two digits shorter than the other.

This algorithm is easily extended to any number representation system (binary, $d$-ary, $e$-ary, etc.) and also to the comparison of $n$ rational numbers.

2. Results

The functions $U(x) = \{1/x\}$ and $\hat{U}(x) = \{(1/x)\}$ (where $\{(y)\}$ stands for the distance to the nearest integer from $y$) are maps of classical continued fractions and extend continued fractions to real numbers. Under a uniform probabilistic model (over the set of all inputs), there is an integral equation of the shape $[0, \alpha]$, the number $L$ of it rations $n \leq d$ to compare two numbers satisfy $P(L \geq k + 1) = \sum_{h=k}^{\infty} h(h(0) - h(\alpha)) \approx \frac{h(0) - h(\alpha)}{2}$, and the moment sums of order $l$ satisfy $\rho(l) = \sum_h h(h(0) - h(\alpha))^l$. The moment sums are related to the branch of $U$, which appears in the linear representation of a symbolic function [6], thus:

**Theorem 1.** The expected cost of the basic $(\hat{\rho})$ and centroid $(\hat{\rho})$ comparison algorithms are expressible as sums over lattice points in $\mathbb{N}$

$$\hat{\rho}(l) = 1 + \frac{1}{2l} + \frac{2}{l(2l)} \sum_{d < c < d} \frac{1}{c^l} \quad \text{and} \quad \hat{\rho}(l) = \frac{2l}{l(2l)} \sum_{d < c < d^2} \frac{1}{c^l} \quad (\phi = (1 + \sqrt{5})/2).$$
With the help of double zeta valued functions (also known as Euler–Ragged r sums), d find d as
\[
\zeta_{+}(s, t) = \sum_{n=1}^{\infty} \sum_{q=1}^{n-1} \frac{1}{n^s q^t}, \quad \text{and} \quad \zeta_{-}(s, t) = \sum_{n=1}^{\infty} \sum_{q=1}^{n-1} \frac{(-1)^n}{n^s q^t},
\]

it is possible to write \( \overline{\rho}() \) as a polynomial valued function in terms of double zeta values as
\[
\overline{\rho}(n) = n \sum_{l=1}^{n-1} (-1)^{l-1} \left( \begin{array}{c} n-1 \\ l \end{array} \right) \overline{\rho}(l+1) = K_0 \ln n + K_1 n + Q(\ln n) + O(1),
\]

where \( K_0 \) is Levy's entropy constant and \( K_1 \) is a Porter-like constant (see [4]):
\[
K_0 = \frac{6 \ln 2}{4} \quad \text{and} \quad K_1 = 18 \frac{\ln 2}{4} + 9 \frac{\ln 2}{4} - \frac{\ln 2}{4} - \frac{1}{2}.
\]
The function \( Q(n) \) is an oscillating function with mean value 0 that satisfies \( Q(n) = O(n^{\delta}) \), where \( \delta \) is any number such that \( \delta > \sup \{ \Re(s) \mid \zeta(s) \neq 0 \} \).

For more details on Flajolet's algol and Vallee's article s, available on the website http://algo.inria.fr/flajolet and http://www.info.unicaen.fr/~brigitte.

References: