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Routing Permutations on Trees

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Summary by Dominique Gouyou-Beauchamps

Abstract

We study the problem of routing permutations on trees. We show that this problem is NP-hard but that it is 5/3-approximable. For a linear network or for a star tree network, the problem is polynomial and we give its average complexity. We extend these results and obtain an upper bound for arbitrary trees. This talk is based on a joint work with Mario Valencia-Pabon, Danièle Gardy, Dominique Barth, and Alain Denise [4].

1. Introduction

The routing problem on communication networks consists in the efficient allocation of resources to connection requests. In the case of all-optical networks, data is transmitted on lightwaves through optical fiber, and several signals can be transmitted through a fiber link simultaneously provided that different wavelengths are used in order to prevent interferences [3]. As the number of wavelengths is a limited resource, it is desirable to establish a given set of connection requests with a minimum number of wavelengths. Then the routing problem for all-optical networks can be viewed as a path coloring problem: it consists in finding a desirable collection of paths on the network associated with the collection of connection requests in order to minimize the number of colors needed to color these paths in such a way that any two different paths sharing a same link are assigned different colors. For simple networks, such as trees, the routing problem is simpler, as there is a unique path for each communication request.

Clearly, such a routing problem can be modeled as a permutation-path coloring problem on trees. An instance of the permutation-path coloring problem on trees is given by a directed symmetric tree graph T on n nodes and a permutation σ of the node set of T. Moreover, we associate with each pair $\big(i,\sigma(i)\big)$, $i \neq \sigma(i)$, $1 \leq i \leq n$, the unique directed path on T from node i to node $\sigma(i)$. Thus, the permutation-path coloring problem for this instance consists in assigning the minimum number of colors to such a permutation-set of paths in such a way that any two paths sharing a same arc of the tree are assigned different colors.

2. Definitions

We model the tree network as a rooted labeled symmetric directed tree $T = (V, A)$ on n vertices, where processors and switches are vertices and links are modeled by two arcs in opposite directions. Let P be a collection of directed paths on T. We assume that the vertices of T are arbitrarily labeled by different integers $\{1, 2, \ldots, n\}$ and that the vertex labeled n is the root vertex of T. We denote $i \rightsquigarrow j$ the unique directed path from vertex i to vertex j. The arc from vertex i to its father (resp.

from the father of i to i), $1 \leq i \leq n-1$, is labeled by i^+ (resp. i^-). We call $T(i)$ the subtree of T rooted at vertex i, $1 \leq i \leq n$.

For any i, $1 \leq i \leq n-1$, the load of an arc i^+ (resp. i^-) of T, denoted by $L_T(P, i^+)$ (resp. $L_T(P, i^-)$), is the number of paths in P using such an arc, and the maximum load among all arcs of T is denoted by $L_T(P)$. We call the coloring number and we denote by $R_T(P)$, the minimum number of colors needed to color the paths in P such that any two paths sharing a same arc in T are assigned different colors. Trivially, we have that $R_T(P) \geq L_T(P)$.

We say that P is a permutation-path set on T if P represents a permutation $\sigma \in S_n$ of the vertex set of T, where $\sigma(i) = j$, $i \neq j$, if and only if $i \rightsquigarrow j \in P$. In the sequel we talk indifferently of a permutation-path set P or of the permutation $\sigma \in S_n$ that P represents. Thus, given a permutation $\sigma \in S_n$ and a tree T on n vertices, the load of the arc i^+, resp. i^-, $1 \leq i \leq n-1$, can be expressed by $L_T(\sigma, i^+) = \left|\{\, j \in T(i) \mid \sigma(j) \notin T(i) \,\}\right|$, resp. $L_T(\sigma, i^-) = \left|\{\, j \notin T(i) \mid \sigma(j) \in T(i) \,\}\right|$.

Let T be a tree on n vertices. The average load of all permutations $\sigma \in S_n$ on T, denoted by $\bar{L}_T$, is defined as $\bar{L}_T = (n!)^{-1} \sum_{\sigma \in S_n} L_T(\sigma)$.

Proposition 1 ([7]). *There is a polynomial time algorithm to color any collection P of paths on any tree such that $L_T(P) \leq R_T(P) \leq \lceil (5/3) L_T(P) \rceil$.*

Let T be a tree on n vertices. We denote by $\bar{R}_T$ the average number of colors needed to color all permutations in S_n on T.

Proposition 2. *Let T be a tree on n vertices. Then $\bar{L}_T(P) \leq \bar{R}_T(P) \leq (5/3)\bar{L}_T(P) + 1$.*

Let T be a tree on $2n$ vertices. We denote by $\tilde{R}_T$ the average number of colors needed to color all involutions in I_{2n} on T.

Proposition 3. *Let T be a tree on $2n$ vertices and let $\tilde{L}_T$ be the average load of all involutions in I_{2n} on T. Then $\tilde{L}_T \leq \tilde{R}_T \leq (3/2)\tilde{L}_T$.*

3. Complexity of Computing the Coloring Number

We show the NP-hardness of the symmetric-path coloring problem on binary trees, answering an open question in [2]. For this, we use a reduction similar to the one used in [6, 10] for proving the NP-hardness of the general path coloring problem on binary trees. We extend this reduction to obtain NP-hardness results on very restrictive instances like involutions on both binary trees and trees having only two vertices with degrees greater than two.

Theorem 1. *Let T be a directed symmetric tree and let P be a collection of directed paths on T. Then, computing $R_T(P)$ is NP-hard in the following cases:*

- *T is a binary tree and P is a collection of symmetric paths on T.*
- *T is a binary tree and P represents an involution of the vertices of T.*
- *T is a tree with maximum degree greater or equal to 4, and P represents a circular permutation of the vertices of T.*
- *T is a tree having only two degrees greater than two and P represents an involution of the vertices of T.*

4. A Lower Bound for the Average Coloring Number

Let $G = (V, A)$ be a directed symmetric graph on n vertices and r a routing function in G which assigns a set of paths on G to route any permutation $\sigma \in S_n$. Let $\bar{L}_{G,r}$ be the average load of all permutations in S_n induced by the routing function r, and let $U \subseteq V$ be a subset of the vertex set of G. We denote by $c(U)$ the cut $(U, \bar{U})$, i.e., the set of arcs $\{\, (u,v) \in A \mid u \in U,\ v \in V \setminus U \,\}$.

Proposition 4. *For any graph $G = (V, A)$ on n vertices, and any routing function r in G,*

$$\bar{L}_{G,r} \geq \frac{1}{n} \max_{U \subseteq V} \left(\frac{|U|(n - |U|)}{|c(U)|} \right).$$

Let T be a tree on n vertices. By the previous proposition, we can deduce that the average load of any arc i^+ of T, $1 \leq i \leq n-1$, denoted by $\bar{L}_T(i)$, satisfies $\bar{L}_T(i) = |T(i)|\bigl(n - |T(i)|\bigr)/n$. Moreover, for any vertex i of T, let $v_T(i) = |T(i)|/n$ and $\tilde{v}_T(i) = \min\bigl(v_T(i), 1 - v_T(i)\bigr)$. Let $\tilde{v}_T = \max_i \tilde{v}_T(i)$.

Proposition 5. *Both inequalities $\bar{L}_T \geq n\tilde{v}_T(1 - \tilde{v}_T)$ and $\bar{R}_T \geq n\tilde{v}_T(1 - \tilde{v}_T)$ hold.*

5. Average Coloring Number on Linear Networks

The main result is the following:

Theorem 2. *The average coloring number of the permutations in S_n to be routed on a linear network on n vertices is $n/4 + (\lambda/2)n^{1/3} + O(n^{1/6})$ where $\lambda = 0.99615\ldots$*

To prove this result, we use enumerative and asymptotic combinatorial techniques (Theorems 3 and 4 below and results of Louchard [12] and Daniels and Skyrme [5]). Our approach uses the same methodology as Lagarias et al. [11] who studied involutions with no fixed point routed on the linear network.

Let W_n be the set of Motzkin walks of length n labeled as follows:

- each South-East step of height i is labeled by an integer between 1 and $(i+1)^2$,
- each East step of height i is labeled by an integer between 1 and $2i+1$.

Theorem 3. [9] *There is a one-to-one correspondence between the elements W_n and those of S_n.*

We use Biane's bijection [1] because it preserves the height of our objects, i.e., the height of a labeled Motzkin walks is equal to the height of the corresponding permutation. Moreover, the height of a permutation is equal to its load.

Let $S_{n,\leq k}$ be the number of permutations in S_n of height at most k and let $S_{n,k}$ be the number of permutations in S_n of height exactly k.

Theorem 4. [8, 13] *We have the identities* $H_k(z) = \sum_{n \geq 0} \sum_{\sigma \in S_{n,k}} z^n = \dfrac{(k!)^2 z^{2k}}{P^\star_{k+1}(z) P^\star_k(z)}$ *and*

$$H_{\leq k}(z) = \sum_{n \geq 0} \sum_{\sigma \in S_{n,\leq k}} z^n = \cfrac{1}{1 - \cfrac{z^2}{1 - 3z - \cfrac{4z^2}{1 - 5z - \cfrac{\vdots}{1 - (2k-1)z - \cfrac{k^2 z^2}{1 - (2k+1)z}}}}},$$

with $P_0(z) = 1$, $P_1(z) = z - 1$ and $P_{n+1}(z) = (z - 2n - 1)P_n(z) - n^2 P_{n-1}(z)$ for $n \geq 1$, where $P^\star$ is the reciprocal polynomial of P, that is $P^\star_n(z) = z^n P_n(1/z)$ for $n \geq 0$.

6. Average Coloring Number on Arbitrary Tree Networks

We can extend the average complexity results on linear networks to arbitrary tree networks.

Theorem 5. *The average load induced by all permutations of S_n on T is $\bar{L}_T = n\tilde{v}_T(1 - \tilde{v}_T) + O(n^{1/2})$.*

Theorem 6. *For all ϵ, there exists $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$ and any tree T on n vertices, the average number of colors $\bar{R}_T$ needed to color any permutation $\sigma \in S_n$ on T satisfies $\bar{R}_T \leq (5/3 + \epsilon)n\tilde{v}_T(1 - \tilde{v}_T)$.*

Let $ST(n)$ denote the directed symmetric star graph on n vertices (i.e., the tree having only one internal vertex connected to $n - 1$ leaves). We call generalized star graph that we denote by $GST(\lambda)$, a directed symmetric tree on n vertices having k branches connected to each other by one vertex, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of the integer $n - 1$ into k parts ($k > 2$) and where λ_i denotes the length of the ith branch (i.e., a branch of length λ_i is a path graph on $\lambda_i + 1$ vertices). We can also obtain the same type of results for generalized star trees and involutions instead of permutations.

Theorem 7. *Let k be a fixed integer greater than 2. The average number of colors needed to color any permutation $\sigma \in S_{nk+1}$ on a generalized star tree $GST(\lambda)$ having $nk+1$ vertices and k branches of length n is $n(k-1)/k + O(n^{1/2})$.*

Theorem 8. *Let T be a tree on $2n$ vertices. The average load induced by all involutions with no fixed points $\sigma \in I_{2n}$ on T is $\bar{L}_T = 2n\tilde{v}_T(1 - \tilde{v}_T) + O(n^{1/2})$.*

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