Algorithms Seminar 1999–2000,
F. Chyzak (ed.), INRIA, (2000), pp. 65–70.

Available online at the URL
http://algo.inria.fr/seminars/.

Difference Equations with Hypergeometric Coefficients

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March 3, 2000

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Abstract

Let k be a difference field with automorphism σ. Let b be an element of k, and L be a linear ordinary difference operator with coefficients in k. A classical problem in the theory of difference equations is to compute all the solutions in k of the equation $L(y) = b$. If C denotes a constant field and if $k = C(n)$ and $\sigma n = n + 1$ or $\sigma n = qn$, there are known algorithms (see [2] for example). Manuel Bronstein presents here a generalization to monomial extensions of $C(n)$ (see [5] for details and generalization).

1. Historical Context

The rational solutions of linear differential equations (equations of the form $\sum_{i=0}^{n} a_i y^{(i)}$) have been first studied a long time ago, for example by Beke and Schlesinger at the end of the last century. In the middle of this century, R. H. Risch gave an algorithm to compute elementary integrals (see [11, 12, 13]). In [8], M. Karr considered difference equations (equations of the form $\sum_{i=0}^{n} a_i y(x+i)$). The link between the linear differential equations and the linear difference equations is now clear, and in [1], an algorithm to compute the rational solutions of this two types of equations with coefficients in $C(x)$ is given. In [2], the author extends the previous algorithm to q-linear difference equations (equations of the form $\sum_{i=0}^{n} a_i y(q^i x)$).

Algorithms to compute the rational solutions of linear differential, difference and q-difference equations with coefficients in $C(x)$ are now available, and extensions of $C(x)$ have been considered. In [14], M. F. Singer gives an algorithm to compute the rational solutions of linear differential equations with coefficients in almost all the Liouvillian extensions of $C(x)$, i.e., the extensions built up using integral, exponential of integral, and algebraic functions. In [7], the authors improve the algorithm for the rational solutions of linear differential equations with coefficients in an exponential extension of $C(x)$. In [6], M. Bronstein adapts the algorithm given in [1] to monomial extensions, and in [5], the author uses the methods given in [2, 6] to find the solutions of linear difference equations in their coefficient field.

2. Introduction

In [6], the author introduced the splitting factorization: he decomposed a polynomial in two factors, the *normal* part where every irreducible factor is coprime with its derivative, and the *special* part where every irreducible factor divides its derivative. He then gave an algorithm to compute the normal part of the denominator of rational solutions of a linear differential equation with coefficients in a monomial extension. In [2], S. Abramov proposed an algorithm to compute a polynomial which is divisible by the denominator of any rational solution of a linear difference equation with coefficients in $C(n)$, where $\sigma n = n+1$ or $\sigma n = qn$. In [7], a method to compute the numerator of the rational solution of a linear differential equation with coefficients in an exponential extension of $C(x)$ is given. Manuel Bronstein now considers difference equations with hypergeometric terms in the coefficients (a term $h(n)$ is hypergeometric if $h(n+1)/h(n)$ is in $C(n)$). He adapts the previous methods to difference equations with coefficients in an hypergeometric extension of $C(n)$, and this gives an efficient algorithm to compute the rational solutions of such equations. Remark that an algorithm to compute the hypergeometric solutions of linear difference equation with coefficients in $C(n)$ is given in [10] and in [4] for q-hypergeometric solutions of q-difference equations.

3. Difference Equations and Hypergeometric Extensions

Let R be a commutative ring of characteristic 0. Let σ be an automorphism of R. Define

- $R_\sigma = \{\, x \in R \text{ such that } \sigma x = x \,\}$ (the set of invariant elements of R);
- $R_{\sigma^\star} = \{\, x \in R \text{ such that } \sigma^n x = x \text{ for some } n > 0 \,\}$ (the set of periodic elements);
- $R^\sigma = \{\, x \in R \text{ such that } \sigma x = ux \text{ for some } u \in R^\star \,\}$ (the set of semi-invariant elements);
- $R^{\sigma^\star} = \{\, x \in R \text{ such that } \sigma^n x = ux \text{ for some } n > 0,\ u \in R^\star \,\}$ (the set of semi-periodic elements).

It is clear that we have the inclusion $R_\sigma \subseteq R^\sigma \subseteq R^{\sigma^\star}$. If R is a unique factorization domain then $R^{\sigma^\star}$ is closed under taking factors, i.e., for any polynomial q in $R^{\sigma^\star}$, each factor p of q is in $R^{\sigma^\star}$. This property is false for R_σ and R^σ, as shown by the example $R = \mathbb{Q}[t]$ and $\sigma(t) = 1-t$: $\sigma(1-t) = t$ and $\sigma(t-t^2) = t-t^2$ is in R_σ (and then in R^σ and in $R^{\sigma^\star}$), whereas t and $1-t$ are in $R^{\sigma^\star}$, but neither in R^σ nor in R_σ.

3.1. Monomial extensions.

Let k be a difference field with automorphism σ. Let (K,σ) be an extension of (k,σ).

Definition 1. t in K is a monomial over k if t is transcendental over k with σt in $k[t]$.

Let σ be an automorphism of K such that $\sigma(t)$ is in $k[t]$. Then σ induces an automorphism of $k(t)$, an automorphism of $k[t]$, and thus $\sigma(t) = at+b$ for some a in $k^\star$ and b in k

Proposition 1 ([9]). *If for all w in $k^\star$ we have $\sigma w \neq aw+b$, then t is transcendental over k and the following equalities hold: $k(t)_\sigma = k_\sigma$ and $k[t]^\sigma = k[t]^{\sigma^\star} = k$.*

3.2. Hypergeometric extensions.

Let σ be such that $\sigma t = at$ for some $a \in k^\star$.

Proposition 2 ([9]). *If for all w in $k^\star$ and $n > 0$ we have $\sigma w \neq a^n w$, then t is transcendental over k and the following equalities hold: $k(t)_\sigma = k_\sigma$ and $k[t]^\sigma = k[t]^{\sigma^\star} = \{\, ct^m \mid c \in k,\ m \geq 0 \,\}$.*

For example, in $C[n]$, let σ be such that $\sigma n = qn$ for some $q \in C^\star$. The property holds whenever q is not a root of unity. Or we can consider $C[n,t]$, with σ such that $\sigma_{|C} = id_C$, $\sigma n = n+1$ and $\sigma t = (n+1)t$; in other words t represents $n!$.

4. Dispersion

Definition 2. Let K be a field of characteristic 0. Let $\phi : K[X] \to K[X]$ be a function. Let p and q be non-zero polynomials in $K[X]$. One defines

- the spread of p and q with respect to ϕ:

$$\mathrm{Spr}_\phi(\mathrm{p},\mathrm{q}) = \{\, m \geq 0 \text{ such that } p \text{ and } \phi^m q \text{ have a non trivial gcd} \,\}$$

– the dispersion of p and q with respect to ϕ:

$$\mathrm{Dis}_\phi(\mathrm{p},\mathrm{q}) = \begin{cases} -1 & \text{if } \mathrm{Spr}_\phi(\mathrm{p},\mathrm{q}) \text{ is empty;} \\ \max(\mathrm{Spr}(p,q)) & \text{if } \mathrm{Spr}_\phi(p,q) \text{ is a finite nonempty set;} \\ +\infty & \text{if } \mathrm{Spr}_\phi(\mathrm{p},\mathrm{q}) \text{ is an infinite set.} \end{cases}$$

These definitions are specialized to the case $p = q$: $\mathrm{Spr}_\phi(p) = \mathrm{Spr}_\phi(p,p)$ and $\mathrm{Dis}_\phi(p) = \mathrm{Dis}_\phi(p,p)$.

Examples are:

– $\mathrm{Dis}_{d/dx}\big(p(x)\big)$ is the maximum of the multiplicity of a root of p minus 1;
– $\mathrm{Spr}_{n\to n+1}\big(p(n)\big)$ is finite (and then $\mathrm{Dis}_{n\to n+1}\big(p(n)\big) < +\infty$);
– $\mathrm{Dis}_{n\to qn}(n)$ is infinite.

Let σ be an automorphism of $k[t]$ such that $\sigma k \subseteq k$. Then the dispersion $\mathrm{Dis}_\sigma(q)$ is infinite if and only if there exists p in $k[t]^{\sigma^\star}\backslash k$ such that p divides q. Also, the dispersion $\mathrm{Dis}_\sigma(h,q)$ is infinite if and only if there exists p in $k[t]^{\sigma^\star}\backslash k$ such that p divides q and $\sigma^n p$ divides h.

Example. Let $a = 2n^7 + 19n^6 + 63n^5 + 81n^4 + 27n^3$ be in $\mathbb{Q}[n]$ and ϕ be the automorphism of $\mathbb{Q}[n]$ over $\mathbb{Q}$ that maps n to $n+1$. The resultant of a and $\phi^m a$ is

$$4m^{19}(2m+5)^3(2m+1)^3(2m-1)^3(2m-5)^3(m-3)^9(m+3)^9,$$

implying that $\mathrm{Spr}_\phi(a) = \{0,3\}$ and $\mathrm{Dis}_\phi(a) = 3$

4.1. **Splitting factorization.** One now extends the splitting factorization of polynomials to difference field: let q in $k[t]$ be decomposed into two factors $q = q_\infty \overline{q}$ such that

– the gcd of q_∞ and $\overline{q}$ is equal to 1,
– for all irreducible factor p of q, p divides q_∞ if p is in $k[t]^{\sigma^\star}$,
– and for all irreducible factor p of q, p divides $\overline{q}$ if p is not in $k[t]^{\sigma^\star}$.

The polynomial q_∞ is the *infinite part* of q, and $\overline{q}$ is its *finite part*. We note that the dispersion $\mathrm{Dis}_\sigma(\overline{q})$ is finite, the dispersion $\mathrm{Dis}_\sigma(q_\infty)$ is infinite, and for all h the dispersion $\mathrm{Dis}_\sigma(h,\overline{q})$ is finite.

4.2. **σ-Orbits.** Given α and β in a field K, the problem of the orbit is to find $m \geq 0$ such that $\alpha^m = \beta$. A bound for the smallest m such that $\alpha^m = \beta$ is given in [3]. The main ideas are as follows: if there exists d such that $\alpha^d = 1$ then one can test whether $\alpha^i = \beta$ for $0 \leq i \leq d$. If it is not the case, then the orbit problem has no solution, otherwise its solutions consist of all the integers of the form $i_0 + kd_0$ where $k \geq 0$, i_0 is the smallest $i \geq 0$ such that $\alpha^i = \beta$ and d_0 is the smallest $d > 0$ such that $\alpha^d = 1$. One can now assume that α is not a root of unity, which implies that the orbit problem has at most one solution. If α is transcendental over $\mathbb{Q}$, the orbit problem has a solution if and only if β is algebraic over $\mathbb{Q}(\alpha)$. Looking at the degree at which α appears in β gives at most one candidate solution for the orbit problem. One can now assume that α is algebraic over $\mathbb{Q}$. This generalizes to find $m \geq 0$ such that $\alpha^{m,\sigma} = \alpha(\sigma\alpha)\dots(\sigma^{m-1}\alpha) = \beta$ (see [3]).

4.3. **Computation of the dispersion.** Let $\sigma : K[X] \to K[X]$ be an automorphism such that $\sigma K \subseteq K$. Then

$$\mathrm{Spr}_\sigma\Big(\prod_i p_i^{e_i}, \prod_j q_j^{f_j}\Big) = \bigcup_{i,j} \mathrm{Spr}_\sigma(p_i,q_j) \quad \text{and} \quad \mathrm{Dis}_\sigma\Big(\prod_i p_i^{e_i}, \prod_j q_j^{f_j}\Big) = \max_{i,j} \mathrm{Dis}_\sigma(p_i,q_j).$$

The computation of the dispersion reduces to the computation of the dispersion of two irreducible polynomials.

Let p and q be irreducible polynomials. Let m be in $\mathrm{Spr}_\sigma(p,q)$. This means that the greatest common divisor of p and $\sigma^m q$ is not trivial. The polynomials being irreducible, this is equivalent to the existence of u in $K^\star$ such that $\sigma^m q = up$. This implies that $\deg p = \deg q$. One just has to consider irreducible polynomials with common degree.

Let p and q be monic irreducible polynomials of $k[t]$ with degree n: $p = t^n + \sum_{i=0}^{n-1} p_i t^i$ and $q = t^n + \sum_{i=0}^{n-1} q_i t^i$. Assume that $\sigma t = at$ for some $a \in k^\star$. Then m is in $\mathrm{Spr}_\sigma(p,q)$ implies $\alpha_i^{m,\sigma} = \beta_i$ for all i such that $p_i q_i \neq 0$, where $\beta_i = q_i/p_i$ and $\alpha_i = a^{n-i} q_i/\sigma q_i$. Therefore, if $\mathrm{Spr}_\sigma(p,q)$ is not empty then p_i and q_i vanish simultaneously. If $p = q = t$ then $\mathrm{Dis}(p,q) = +\infty$. Otherwise, this reduces to the orbit problem $\alpha^{m,\sigma} = \beta$ for α, β in $k^\star$ and m in $\mathrm{Spr}(p,q)$. Remark that if $\sigma w \neq a^d w$ for all w in $k^\star$ and $d > 0$ then $\alpha^d \neq 1$ for all $d > 0$. So, the orbit problem has at most one solution and then $\mathrm{Spr}_\sigma(p,q)$ has at most one element.

One can extend the computation of the dispersion to rational functions: let $f = p/q$ with relatively prime p and q in $C[n]$. Let $\mathrm{Dis}_\sigma(f) = \max\big(\mathrm{Dis}_\sigma(p), \mathrm{Dis}_\sigma(p,q), \mathrm{Dis}_\sigma(q,p), \mathrm{Dis}_\sigma(q)\big)$ and $\nu_\infty(f) = \deg q - \deg p$. Then $\nu_\infty(f^{m,\sigma}) = m\nu_\infty(f)$. And if f is not in C then $\mathrm{Dis}_\sigma(f^{m,\sigma}) = \mathrm{Dis}_\sigma(f) + m - 1$.

This last equality allows us to reduce orbit problems to dispersions whenever α is not constant.

5. Rational Solutions of Difference Equations

Let t be a monomial over $k = C(n)$. Let σ be such that $\sigma n = n + 1$ and $\sigma t = at$ for some a in k such that $\sigma w \neq a^d w$, for all w in $k^\star$ and $d > 0$. Let $L = \sum_{i=0}^{N} a_i \sigma^i$ be a linear difference operator, with the a_i's in $k[t]$ and both a_0 and a_N not equal to 0. Let b be in $k[t]$. The aim of this section is to described an algorithm to find y in $k(t)$ such that $L(y) = b$ (if there exists such a y).

5.1. **Denominator of a rational solution.** The first problem is to find a bound for the finite part of any y in $k(t)$ such that $L(y) = b$. This means to compute a polynomial q in $k[t]$ such that if $L(y) = b$ then $yq = p/d_\infty$ where p is in $k[t]$ and d_∞ in $k[t]^{\sigma^\star}$. We outline the ideas here, proofs and technical details are given in [5].

Let a_0 be decomposed: $a_0 = a_{0,\infty}\overline{a_0}$. Let y be in $k(t)$ such that $L(y) = b$, where $y = p/d$ and $d = d_\infty \overline{d}$. Then $\mathrm{Dis}_\sigma(\overline{d}) \leq \max\big(-1, \mathrm{Dis}_\sigma(a_N, \overline{a_0}) - N\big)$. Let $h > 0$ be an integer. One can compute an operator $L_h = b_s \sigma^{sh} + b_{s-1}\sigma^{(s-1)h} + \cdots + b_0$ such that $L_h = RL$ for some R in $k(t)[\sigma]$. It follows that $L_h(y) = Rb$ for any b in $k[t]$ and any solution y in $k(t)$ of $L(y) = b$. We get that every solution y in $k(t)$ of $L(y) = b$ satisfies an equation of the form

$$c_s \sigma^{hs}(y) + \cdots + c_1 \sigma^h(y) = d_h$$

where $c_0, \ldots, c_s, d_h$ are in $k[t]$ and $c_s \neq 0$. If h was chosen such that $\mathrm{Dis}_\sigma(\overline{d}) < h$ then $\overline{d}$ divides $\gcd_{0 \leq i \leq s}(\sigma^{-ih} c_i)$. This gives us a polynomial q such that if $L(y) = b$ then $qy = p/d_\infty$ with p in $k[t]$ and d_∞ in $k[t]^{\sigma^\star}$

Example. Consider $y(n+2) - (n! + n)y(n+1) + n(n! - 1)y(n) = 0$. If we define σ by $\sigma n = n + 1$ and $\sigma t = (n+1)t$ then the associated difference operator is $\sigma^2 - (t+n)\sigma + n(t-1)$. $a_N = 1$, $a_0 = \overline{a_0} = n(t-1)$ and $\mathrm{Dis}_\sigma(a_N, \overline{a_0}) = -1$. Then $\mathrm{Dis}_\sigma(\overline{d}) \leq -1$ and $\overline{d} \in C(n)$. So, if there exists $y \in C(n)(t)$ such that $L(y) = b$ then y is in $C(n)[t, t^{-1}]$.

Remark. The same results holds for the q-difference equation: let q be transcendental over $\mathbb{Q}$. Let σ be such that $\sigma x = qx$. Consider the q-difference equation

$$q^3(qx+1)y(q^2x) - 2q^2(x+1)y(qx) + (x+q)y(x) = 0 \qquad (1)$$

We have $\overline{a_0} = x + q$, $a_2 = q^3(qx+1)$. The resultant of a_2 and $\sigma^m(\overline{a_0})$ is $q^3(q^2 - q^m)$, which implies that $\mathrm{Dis}_\sigma(a_2, \overline{a_0}) = 2$ hence that any solution of (1) has a denominator of the form $x^n\overline{d}$ where $\mathrm{Dis}_\sigma(\overline{d}) \leq 0$. Using the bound $h = 1$, we get $L_h = L$ and $\overline{d}$ divides the greatest common divisor of $\gcd_{0 \leq i \leq 2}(\sigma^{-i} a_i) = \gcd\big(x+q, \sigma^{-1}(q^2(x+1)), \sigma^{-2}(q^3(qx+1))\big) = \gcd\big(x+q, q(x+q), q^2(x+q)\big) = x+q$.

Therefore, any rational solution of (1) can be written as $y = p/\big(x^n(x+q)\big)$ where $n \geq 0$ and p is in $\mathbb{Q}[x]$.

The indicial equation at $x = 0$ is $qZ^2 - 2q^2Z + q^3 = 0$ (see [2]). Its only solution of the form $Z = q^n$ is for $n = 1$, which implies that any rational solution of (1) can be written as $y = p/\big(x(x+q)\big)$. Replacing y by this form, we get $p(q^2x) - 2p(qx) + p(x) = 0$ (whose solution space is $\mathbb{Q}(q)$, which implies that the general rational solution of (1) is $y = C/\big(x(x+q)\big)$ for any C in $\mathbb{Q}(q)$).

5.2. **Laurent polynomial solution.** The problem of finding rational solutions y of $L(y) = b$ is reduced to finding y in $k[t, t^{-1}]$ such that $L(y) = b$, where b is in $k[t, t^{-1}]$ and $L = \sum_{i=0}^{N} a_i \sigma^i$ is a difference operator, with $a_i \in k[t]$ and non-zero a_0 and a_N. This decomposes in two steps:

1. find a bound for the degree and the order in t of y;
2. compute the coefficients of y, seen as a Laurent polynomial in t.

5.2.1. *Bound for the degree and order of a polynomial solution.* One rewrites L as $\sum_{j=\nu}^{d} t^j L_j$ where the L_j's are in $k[\sigma]$ and L_ν and L_d are not equal to zero. Let $y = y_\delta t^\delta + \cdots + y_\gamma t^\gamma$ be in $k[t, t^{-1}]$ for integers γ and δ satisfying $\gamma \geq \delta$ and such that neither y_δ nor y_γ is equal to zero. Let b be in $k[t, t^{-1}]$. If $L(y) = b$, then

1. either $\delta \geq \nu(b) - \nu$, or $L_\nu(y_\delta t^\delta) = 0$;
2. either $\gamma \leq \deg b - d$, or $L_d(y_\gamma t^\gamma) = 0$.

The problem is reduced to considering difference operators $T = \sum_{i=m}^{M} A_i \sigma^i$ with $A_i \in C[n]$ for non-zero A_m and A_M, and to searching bounds for $\gamma \in \mathbb{Z}$ such that $T(zt^\gamma) = 0$ for some z in $C(n)$. Let $e = -\nu_\infty(\sigma t/t) = \nu_\infty(a)$. There are three possibilities:

- if $e > 0$ then $(\deg_n A_m - \deg_n T)/e \leq \gamma \leq (\deg_n T - \deg_n A_M)/e$;
- if $e < 0$ then $(\deg_n T - \deg_n A_m)/e \leq \gamma \leq (\deg_n A_m - \deg_n T)/e$;
- if $e = 0$ then $\alpha = a(\infty) \in C^\star$. We decompose $A_i = a_{i,\alpha_i} n^{\alpha_i} + \cdots$. We define $Q(z) = \sum_{i|\alpha_i = \max_j(\alpha_j)} a_{i,\alpha_i} z^i$. We have $Q(\alpha^\gamma) = 0$. This problem can be solved if $\alpha^d \neq 1$ for all $d \geq 0$ (see section 4.2).

5.2.2. *Coefficients of a Laurent polynomial solution.* This is a generalization of the specialization given in [7].

We have found γ and δ such that if y is in $k[t, t^{-1}]$ with $L(y) = b$ then $\deg_t(y) \leq \gamma$ and $\mathrm{val}_t(y) \geq \delta$. Let $z = t^\delta y$. Note that $\deg_t(z) \leq \gamma - \delta = J$. One has to consider the problem $L(z) = b$ where L is in $k[t][\sigma]$ and b in $k[t]$. Let $L = \sum_{j=0}^{d} t^j L_j$ with L_j in $k[\sigma]$, and L_0, L_d not equal to zero.

- if $J = 0$ then $L(z) = \sum_{j=0}^{d} t^j (L_j z)$. But $L_j(z)$ is in k so $L(z) = b$ implies $L_j(z) = b_j$ for all j and this reduces to difference equations with coefficients in $C[n]$;
- if $J > 0$ then one decomposes $z = z_0 + t\overline{z}$ where $z_0 = z(0)$ is in $C(n)$. Then $L_0(z_0) = b_0$ and one can find z_0. So, $L(z) = (L - L_0)(z_0) + L(t\overline{z}) + L_0(z_0)$ and $L(z) = b$ implies

$$\begin{aligned} L(t\overline{z}) &= b - b_0 - (L - L_0)z_0 \\ t\tilde{L}(\overline{z}) &= t\left(\frac{b - b_0}{t}\right) - t\frac{(L - L_0)z_0}{t} \end{aligned}$$

  This gives us a new difference equation with a solution $\overline{z}$ of degree strictly less than J. By induction, one can find $\overline{z}$.

Example. Consider $y(n+2) - (n! + n)y(n+1) + n(n! - 1)y(n) = 0$, which is associated to the difference operator

$$L = \sigma^2 - (t + n)\sigma + n(t - 1) = t(n - \sigma) + (\sigma^2 - n\sigma - n) = tL_1 + L_0$$

Using the same notations as previously, $e = -\nu_\infty(\sigma t/t) = -\nu_\infty(n+1) = 1$ and then $y = y_0 + y_1 t$. One first considers $L_0(y_0) = \sigma^2 y_0 - n\sigma y_0 - n y_0 = 0$, and finds that $y_0 = 0$. Then:

$$L(tz_1) = (n+2)(n+1)t\sigma^2(z_1) - (n+1)(t+n)t\sigma(y_1) + n(t-1)ty_1,$$

from which follows that

$$\tilde{L}(y_1) = (n+2)(n+1)\sigma^2(y_1) - (n+1)(t+n)\sigma(y_1) + n(t-1)y_1 = 0.$$

This implies that $y_1 = c/n$. Then $y = y_1 t = (c/n)n! = c(n-1)!$.

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