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Classifying ECO-Systems and Random Walks

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Abstract

This talk presents a classification by rationality, algebraicity or transcendence of ECO-systems (Enumerating Combinatorial Objects) and of more general random walks. It is based on an article by Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy and Dominique Gouyou-Beauchamps [1].

1. Introduction

A *generating tree* is defined by a system (an axiom and a family of rewriting rules)

$$\Big((s_0), \big\{(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k))\big\}_{k\geq 0}\Big). \tag{1}$$

Here, the axiom (s_0) specifies the degree of the root, while the productions $e_i(k)$ (with $e_i(k) > 0$) list the degrees of the k descendants of a node labelled k (note the constraint on the number of descendants of a node). Such a system constitutes an *ECO-System*.

Example. 123-avoiding permutations. Consider the set $\mathfrak{S}_n(123)$ of permutations of length n that *avoid the pattern* 123: there exist no integers $i < j < k$ such that $\sigma(i) < \sigma(j) < \sigma(k)$. For instance, $\sigma = 4213$ belongs to $\mathfrak{S}_4(123)$ but $\sigma = 1324$ does not, since $\sigma(1) < \sigma(3) < \sigma(4)$.

Observe that if $\tau \in \mathfrak{S}_{n+1}(123)$, then the permutation σ obtained by erasing the entry $n+1$ from τ belongs to $\mathfrak{S}_n(123)$. Conversely, for every $\sigma \in \mathfrak{S}_n(123)$, insert the value $n+1$ in each place where this is compatible with the avoiding rule; this gives an element of $\mathfrak{S}_{n+1}(123)$. For example, the permutation $\sigma = 213$ gives 4213, 2413 and 2143, by insertion of 4 in first, second and third place respectively. The permutation 2134, resulting of the insertion of 4 in the last place, does not belong to $\mathfrak{S}_4(123)$. This process can be described by a tree whose nodes are the permutations avoiding 123: the root is 1, and the children of any node σ are the permutations derived as above (see Figure 1(a)).

Let us now label the nodes by their number of children: we obtain the tree of Figure 1(b). It can be proved that the k children of any node labelled k are labelled respectively $k+1, 2, 3, \dots, k$. Thus the tree we have constructed is the generating tree obtained from the following system:

$$\Big((2), \big\{(k) \rightsquigarrow (2)(3)\dots(k-1)(k)(k+1)\big\}_{k\geq 2}\Big). \tag{2}$$

Notations. We assume that all the values appearing in the generating tree are positive.

In the generating tree, let f_n be the number of nodes at level n and s_n the sum of the labels of these nodes. By convention, the root is at level 0, so that $f_0 = 1$. In terms of walks, f_n is the number of walks of length n. The generating function associated to the system is $F(z) = \sum_{n\geq 0} f_n z^n$.

Note that $s_n = f_{n+1}$, and that the sequence $(f_n)_n$ is nondecreasing.

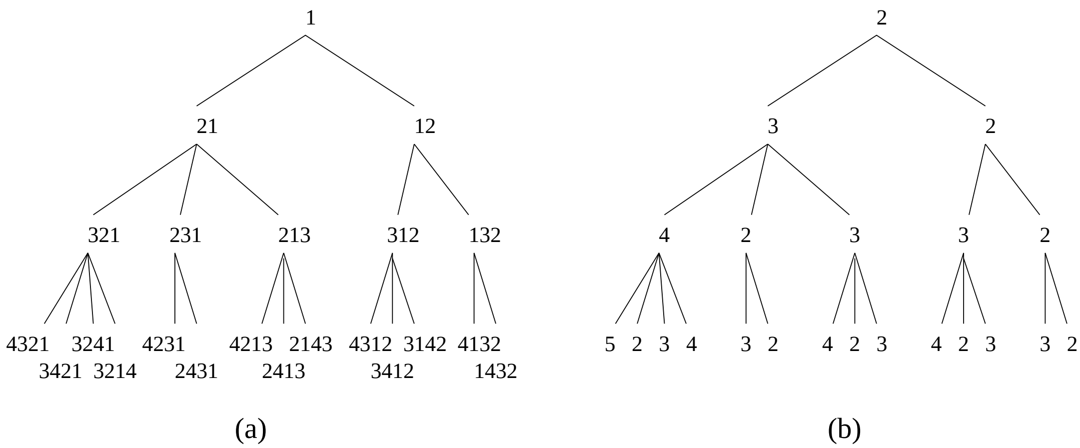

FIGURE 1. The generating tree of 123-avoiding permutations: (a) nodes labelled by the permutations; (b) nodes labelled by the numbers of children.

Now let $f_{n,k}$ be the number of nodes at level n having label k (or the number of walks of length n ending at position k). The following generating functions will be of interest:

$$F_k(z) = \sum_{n\geq 0} f_{n,k} z^n \qquad \text{and} \qquad F(z,u) = \sum_{n,k\geq 0} f_{n,k} z^n u^k.$$

We have $F(z) = F(z,1) = \sum_{k\geq 1} F_k(z)$. Furthermore, the F_k's satisfy the relation

$$F_k(z) = [k = s_0] + z \sum_{j\geq 1} \pi_{j,k} F_j(z), \tag{3}$$

where $[k = s_0]$ is 1 if $k = s_0$ and 0 elsewhere and $\pi_{j,k}$ denotes the number $\big|\{\, i \leq j \mid e_i(j) = k \,\}\big|$ of one-step transitions from j to k. This is equivalent to the recurrence $f_{n+1,k} = \sum_{j\geq 1} \pi_{j,k} f_{n,j}$ for the numbers $f_{n,k}$ (with $f_{0,s_0} = 1$), that results from tracing all the paths that lead to k in $n+1$ steps.

We refer to [1] for random generation using counting and generating trees.

2. Rational Systems

ECO-systems satisfying strong regularity conditions lead to rational generating functions. This covers systems that have a finite number of allowed degrees, as well as systems where the sum of the labels at level k depends linearly on k.

Proposition 1. *If finitely many labels appear in the tree, then $F(z) = F(z,1)$ is rational.*

Proof. Only a finite number of F_k's are nonzero; they are related by linear equations like Equation (3) above and therefore rational. $F(z)$ is a finite sum of these, and is also rational. □

Example. Fibonacci numbers are generated by the system $\big((1), \{(k) \rightsquigarrow (k)^{k-1}((k \bmod 2) + 1)\}\big)$ that can also be written as $\big((1), \{(1) \rightsquigarrow (2), (2) \rightsquigarrow (1)(2)\}\big)$.

Proposition 2. *Let $\sigma(k) = e_1(k) + e_2(k) + \cdots + e_k(k)$. If σ is an affine function of k, say $\sigma(k) = \alpha k + \beta$, then the series $F(z)$ is rational. More precisely:*

$$F(z) = \frac{1 + (s_0 - \alpha)z}{1 - \alpha z - \beta z^2}.$$

Proof. Let $n \geq 0$ and let $k_1, k_2, \ldots, k_{f_n}$ denote the labels of the f_n nodes at level n. Then

$$f_{n+2} = s_{n+1} = (\alpha k_1 + \beta) + (\alpha k_2 + \beta) + \cdots + (\alpha k_{f_n} + \beta) = \alpha s_n + \beta f_n = \alpha f_{n+1} + \beta f_n.$$

We know that $f_0 = 1$ and $f_1 = s_0$. The result follows. □

Example. The system $\big((2), \{(k) \rightsquigarrow (2)^{k-1}(k+1)\}\big)$ produces the Fibonacci numbers of even index.

Proposition 2 can be adapted to apply to systems that "almost" satisfy its criterion (see [1]).

3. Algebraic Systems

Systems where a finite modification of the set $\{1, \ldots, k\}$ is reachable from k lead to algebraic generating functions.

The possible moves from k are given by the rule:

$$(k) \rightsquigarrow \{(0), \ldots, (k-1)\} \setminus \{(k-i) \mid i \in B\} \ \cup\ \{\{(k+j) \mid j \in A\}\}, \tag{4}$$

where $A \subset \mathbb{N}$ and $B \subset \mathbb{N}^+$ are a finite multiset (denoted $\{\{\ldots\}\}$) and a finite set specifying respectively the *allowed forward jumps* (possibly coloured) and the *forbidden backwards jumps*.

Observe that these walk models are not necessarily ECO-systems, first because we allow labels to be zero—but a simple translation can take us back to a model with positive labels—, and second because we do not require (k) to have exactly k successors.

In this section $f_{n,k}$ is the number of walks of length n ending at point k and $f_n(u) = \sum_{k \geq 0} f_{n,k} u^k$ is the coefficient of z^n in $F(z, u)$.

We continue this section with the example $A = \{4, 15\}$ and $B = \{2\}$, axiom (0) and the corresponding family of rules

$$\big\{(k) \rightsquigarrow (0)(1) \ldots (k-3)(k-1)(k+4)(k+15)\big\}.$$

This corresponds in generating functions to substituting u^k in

$$u^0 + \cdots + u^{k-1} - u^{k-2} + u^{k+4} + u^{k+15} = \frac{1-u^k}{1-u} - u^{k-2} + u^{k+4} + u^{k+15}$$

for $k \geq 2$. This gives the recurrence $f_{n+1}(u) = \frac{f_n(1) - f_n(u)}{1-u} + (u^4 + u^{15} - u^{-2}) f_n(u)$, and yields the functional equation

$$F(z,u) = 1 + z\left(\frac{F(z,1) - F(z,u)}{1-u} + P(u)F(z,u) - \{u^{<0}\}\sum_{n \leq 0} z^n L[f_n](u)\right). \tag{5}$$

Here $P(u) = \sum_{\alpha \in A} u^\alpha - \sum_{\beta \in B} u^{-\beta}$ and $L[g](u) = \frac{g(1) - g(u)}{1-u} + P(u)g(u)$. Equation (5) may be rewritten as

$$F(z,u)\left(1 + \frac{z}{1-u} - zP(u)\right) = 1 + \frac{z}{1-u} F(z,1) - z\sum_{j=0}^{b-1} c_j(u) \partial_u^j F(z,0),$$

where the $c_j(u)$ are Laurent polynomials. The kernel $K(z,u)$ of Equation (5) is the coefficient of $F(z,u)$ in the left-hand side of this equation. $F(z,u)K(z,u)$ is a linear combination of $b+1$ unknown functions. Solving $K(z,u) = 0$ in u gives $b+1$ convergent branches $u_i(z)$ which, in turn, give the $\partial_u^j F(z,0)$ through a $(b+1) \times (b+1)$ linear system, and from there $F(z,1)$, which is algebraic.

Proposition 3. *The generating function $F(z,1)$ counting the number of walks, starting from zero and irrespective of their endpoint is algebraic and $F(z,1) = -1/z\prod_{i=0}^{b}(1-u_i)$, where $b = \max B$ and $u_i(z)$ are the finite solutions at $z=0$ of the equation $K(z,u)=0$.*

Examples of algebraic systems are the Catalan numbers $\{(k) \rightsquigarrow (0)(1)\dots(k)(k+1)\}$, the Motzkin numbers $\{(k) \rightsquigarrow (0)\dots(k-1)(k+1)\}$, the Schröder numbers $\{(k) \rightsquigarrow (0)\dots(k-1)(k)(k+1)\}$ or the m-ary trees $\big((m), \{(k) \rightsquigarrow (m)\dots(k)(k+1)(k+2)\dots(k+m-1)\}\big)$.

4. Transcendental Systems

4.1. **Transcendence.** If the coefficients of a series grow too fast, its radius of convergence is zero.

Proposition 4. *Let b be a nonnegative integer. For $k \geq 1$, let $m_k = \big|\{\, i \mid e_i(k) \geq k-b \,\}\big|$. Assume that:*

1. *for all k, there exists a forward jump from k (i.e., $e_i(k) > k$ for some i),*
2. *the sequence $(m_k)_k$ is non-decreasing and tends to infinity.*

Then the generating function of the system has radius of convergence 0.

Proof. See [1]. □

However, there are ECO-systems or walks that are transcendental with positive radius of convergence such as $\{(k) \rightsquigarrow (2)(4)\dots(2k)\}$ or $\{(k) \rightsquigarrow (\lceil k/2 \rceil)^{k-1}(k+1)\}$.

4.2. **Holonomy.** A subclass of transcendental functions is the class of holonomic functions. A series is said to be *holonomic* or *D-finite* if it satisfies a linear differential equation with polynomial coefficients in z. Equivalently, its coefficients f_n satisfy a linear recurrence relation with polynomial coefficients in n. Given a sequence f_n, the OGF (ordinary generating function) $\sum f_n z^n$ is holonomic if and only if the EGF (exponential generating function) $\sum f_n z^n/n!$ is holonomic.

The following table gives examples of holonomic and non-holonomic transcendental systems with references to the Encyclopedia of Integer Sequences (EIS) by Sloane and Plouffe [2, 3].

Axiom	Rewriting rules	Name	EIS Id.	Generating Function
	Holonomic OGF			**EGF**
(1)	$(k) \rightsquigarrow (k+1)^k$	Permutations	M1675	$1/(1-z)$
(2)	$(k) \rightsquigarrow (k)(k+1)^{k-1}$	Arrangements	M1497	$e^z/(1-z)$
(1)	$(k) \rightsquigarrow (k-1)^{k-1}(k+1)$	Involutions	M1221	$e^{z+z^2/2}$
(2)	$(k) \rightsquigarrow (k+1)^{k-1}(k+2)$	Partial permutations	M1795	$e^{z/(1-z)}/(1-z)$
	Nonholonomic OGF			**EGF**
(1)	$(k) \rightsquigarrow (k)^{k-1}(k+1)$	Bell numbers	M1484	e^{e^z-1}
(2)	$(k) \rightsquigarrow (k-1)(k)^{k-2}(k+1)$	Bessel numbers	M1462	—

Bibliography

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[2] Encyclopedia of integer sequences. – Available from `http://www.research.att.com/~njas/sequences/`.

[3] Sloane (N. J. A.) and Plouffe (Simon). – *The encyclopedia of integer sequences.* – Academic Press Inc., San Diego, CA, 1995, xiv+587p.