Concrete Resolution of Differential Problems using Tannakian Categories

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Abstract

Given a linear ODE with polynomial coefficients, one easily finds local information about its solutions. To obtain global information of algebraic nature (operator factorization, explicit finite form, algebraic relations between solutions), one classically reduces the problem to determining rational or exponential solutions of auxiliary linear ODE's. The latter are often uneasy to compute in practice, and we show by a few examples how to advantageously substitute differential systems that are simpler to construct, solve or study.

1. Solving Linear Differential Equations

The main question when studying a linear differential operator L is how to "solve" for its solutions. "Solving", however, covers several meanings. Throughout this text, L denotes a differential operator acting on a function y in the variable x by $L(y) = a_n y^{(n)} + \cdots + a_0 y$ for polynomials a_i in x with coefficients in a field C. This field is $\mathbb{Q}$, $\bar{\mathbb{Q}}$ or $\mathbb{C}$ in practice.

The simplest way to solve is the determination of local information, like a basis of formal solutions in the neighbourhood of 0. The general form of a formal solution is the formal series

$$y = x^{\alpha}(p_0(\ln x) + p_1(\ln x)x^{1/r} + \cdots + p_i(\ln x)x^{i/r} + \cdots)$$

for polynomials p_i with uniformly bounded degrees. Here, r is a positive integer, the ramification, and p_0 is assumed to be non-zero so as to ensure that the highest possible power has been incorporated into the generalized exponent $\alpha \in C[x^{1/r}]$. The power x^{α} is nothing but $\exp \int \alpha/x \, dx$, the formal solution of $y' = (\alpha/x)y$. This approach by generalized exponents is due to Van Hoeij [12] and unifies regular and irregular singular expansions. A similar treatment was developed in the case of systems by Barkatou [1] and Pflügel [5].

Of course, the most generally understood acceptance of "solving" relates to resolution in closed form. By simultaneously considering the bases of formal solutions in the neighbourhood of all possible singularities of the operator L, namely, the zeroes of its leading coefficient $a_n(x)$, several algorithms are available to search for solutions in various classes of closed form, like polynomial solutions $y \in C[x]$, rational solutions $y \in C(x)$, exponential solutions y for which $y'/y \in C(x)$, or liouvillian solutions y for which y'/y is algebraic over $C(x)$. See [4, 13] and the references there.

Note that each solution s in the above classes supplies a first-order right-hand factor of the operator L, namely $\partial - s'/s$ where ∂ denotes the derivation operator with respect to x. A more general problem is that of the factorization of operators from the ring $C(x)[\partial]$ of linear differential operators with rational function coefficients, and the search for higher-order right-hand factors. This relates to differential Galois theory. More specifically, polynomial, rational, and exponential solutions correspond to factorization in this ring, whereas liouvillian solutions correspond to the more complex problem of absolute factorization [13], i.e., factorization of an operator $L \in C(x)[\partial]$ with factors in $K[\partial]$ for an algebraic closure K of $C(x)$. In any case, factorization relates to solving since any solution of any right-hand factor is a solution of the original operator. Furthermore, specialized algorithms exist for linear differential equations of small orders.

Right-hand factors of an operator are a first type of auxiliary operators or lower order that simplify solving. More generally, another form of "solving" the operator L is by looking for its solutions that can be viewed as powers, products, or wronskians of an auxiliary operator, or system of operators, of lower order. This is the main discussion of the next sections. Applications include the classification of solutions, connexion problems, number theory (by looking for differential equations of minimal order), and the search for first integrals of non-linear differential equations.

2. Lower Order Equations and Symmetric Power Solutions

As an example, consider the third-order equation $y''' - 4ry' - 2r'y = 0$ $(r \in C(x))$. It admits a basis of solutions of the form $(z_1 = y_1^2, z_2 = y_2^2, z_3 = y_1 y_2)$, where both y_1 and y_2 are solutions of the same second-order equation $y'' = ry$. To obtain such special solutions of a higher-order operator L, the crucial relation to be used is $z_1 z_2 = z_3^2$. Indeed, considering the formal solution $\tilde{z}_i = x^{\alpha_i}\Sigma_i$ corresponding to the expansion of each actual function z_i, we obtain that the formal expansion of the product $z_1 z_2$ is the product of formal expansions $\tilde{z}_1\tilde{z}_2 = x^{\alpha_1+\alpha_2}\Sigma_1\Sigma_2$. Identifying those generalized exponents for L that can be a sum of two terms therefore supplies a set of candidate exponents for the auxiliary operator and the z_i. Note that the original third-order equation has been replaced by a "simpler system" consisting of a second-order equation and a quadratic relation.

3. Liouvillian Solutions

To solve an operator L for its liouvillian solutions, one looks for the possible irreducible polynomials P of the form $X^m - b_{m-1}X^{m-1} - \cdots - b_0$ such that $P(u) = 0$ implies $L(\exp\int u\,dx) = 0$ [10]. Given the order n of the operator, differential Galois theory shows that only finitely many degrees are possible for the polynomial P. There exists an algorithm to compute the list of the possible numbers m: for $n = 2$, the list is 1, 2, 4, 6, and 12; for $n = 3$, it is 1, 3, 6, 9, 21, and 36 [6, 7, 9]; for $n = 4$ and higher, a formula is known for the maximum number of the list.

By construction, the roots u_i of P are logarithmic derivatives y_i'/y_i of a solution of L, and $b_{m-1} = \sum_i u_i = \sum_i y_i'/y_i$ is the logarithmic derivative of the product $\prod_i y_i$. A necessary and sufficient condition for the existence of a polynomial P of degree m above, which describes the liouvillian solutions of L is that there exists a polynomial of degree m in solutions of L whose logarithmic derivative is rational, and which is the product of linear factors. More specifically, for a solution basis $(z_1, \ldots, z_m)$ of L the product $\prod_i y_i$ is searched for under the form $\prod_i(c_{i,1}z_1 + \cdots + c_{i,m}z_m)$.

The search for liouvillian solutions therefore reduces to the search for exponential solutions. To this end, the present work allows to avoid computing the equation for the symmetric power, which is too large, but prefers a more compact representation.

4. Factorization and Alternate Power Solutions

As another typical example, let us consider the search for a right-hand factor $H = \partial^2 - b_1\partial - b_0$ of order 2 of the operator $L = \partial^4 - a_2\partial^2 - a_1\partial - a_0$ of order 4. For any solution basis (y_1, y_2) of H, the operator H is given by the determinantal representation

$$H(y) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}^{-1} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = y'' - \frac{\omega_{0,2}}{\omega_{0,1}}y' + \frac{\omega_{1,2}}{\omega_{0,1}}y \quad \text{where} \quad \omega_{i,j} = \begin{vmatrix} y_1^{(i)} & y_2^{(i)} \\ y_1^{(j)} & y_2^{(j)} \end{vmatrix}.$$

To obtain a factor of order 2, we now search for an exponential solution and show that it can be interpreted as a determinant $\omega_{0,1}$. Let A be the companion matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_0 & a_1 & a_2 & 0 \end{pmatrix}, \qquad \text{and let} \qquad Y = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}, \qquad \text{so that} \qquad Y' = AY.$$

Let us introduce the vector $Z = (\omega_{0,1}, \omega_{0,2}, \omega_{0,3}, \omega_{1,2}, \omega_{1,3}, \omega_{2,3})^T$. In view of their definition, the $\omega_{i,j}$ satisfy differential relations like $\omega'_{0,1} = \omega_{0,2}$, $\omega'_{0,3} = \omega_{1,3} + a_0\omega_{0,0} + a_1\omega_{0,1} + a_2\omega_{0,2}$, and so on. From them, we find a matrix

$$\Lambda_2(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_0 & 0 & a_2 & 0 & 0 & 1 \\ 0 & -a_0 & 0 & -a_1 & 0 & 0 \end{pmatrix} \qquad \text{such that } Z' = \Lambda_2(A)Z.$$

Again, we then only look for exponential solutions Z of the matrix $\Lambda_2(A)$, which is easy to construct and contains more information than the usual single auxiliary equation used for factorization. Finally, one has to check that the solution Z is a determinant. For this, a necessary and sufficient condition is the Plücker relation, which here simply reduces to $\omega_{0,1}\omega_{2,3} - \omega_{0,2}\omega_{1,3} + \omega_{0,3}\omega_{1,2} = 0$.

To rephrase the method in a more formal way, introduce V, the solution space of L. The search for Z is indeed a search for objects in the 2-exterior power $\Lambda_2(V)$, i.e., the vector space of linear combination of formal 2-exterior products $v \wedge w$, $(v, w) \in V^2$, which satisfy the rule $w \wedge v = -v \wedge w$. Pure exterior product $u \wedge v$ are interpreted as determinants. The search for Z is therefore equivalent to the search for a pure exterior product $\omega_{0,1} \in \Lambda^2(V)$ such that the 1-dimensional vector space $C\omega_{0,1}$ is stable under the action of the differential Galois group of L.

Here the search for a second-order right-hand factor of a fourth-order equation has been reduced to solving a "simpler" system of six first-order equations.

5. Module and Dual Module Associated with an Operator

As an important tool for the study of a linear differential operator L, one classically associates a canonical module in the following way. For L in the algebra $k[\partial]$ of linear differential operators with coefficients in a field k, one considers the quotient $M = k[\partial]/k[\partial]L$ of $k[\partial]$ by its left ideal $k[\partial]L$. The left module M can be viewed as the module $k[\partial]y$ generated by a generic solution y of the operator L. Linear constructs on and between solution spaces of operators, like (direct or usual) sums, (symmetric or exterior or usual commutative) products, (indefinite) integration, and so on, correspond to constructs on and between the corresponding $k[\partial]$-modules.

A variant module is obtained by endowing the dual k-vector space M^* with a $k[\partial]$-module structure. Let r be the order of L, then M is of dimension r and its dual $M^* = \mathrm{Hom}_k(M, k)$ is isomorphic to k^r. Now let A be the companion matrix of L and $(b_1, \ldots, b_r)$ be the canonical basis of M^*. The latter is turned into a $k[\partial]$-module by defining an operator ∇ on M^* by the action

$$(\nabla b_1, \ldots, \nabla b_r)^T = -A^T(b_1, \ldots, b_r)^T$$

and letting ∂ act by ∇. Thus, $\nabla(am) = a\nabla m + a'm$ when $a \in k$ and $m \in M$. From this Leibniz rule applied to the product $y_1b_1 + \cdots + y_rb_r = (y_1, \ldots, y_r)(b_1, \ldots, b_r)^T$, we derive the equality

$$Y' = AY \qquad \text{for} \qquad Y = (y_1, \ldots, y_r)^T$$

whenever $\nabla(y_1 b_1 + \cdots + y_r b_r) = 0$. Note that this $k[\partial]$-module structure on M^* usually does not make it the dual $k[\partial]$-module $\mathrm{Hom}_{k[\partial]}(M, k)$, for the operator L usually has no solution in k.

The modules M^* allow for a better description of the calculations suggested in the previous sections through a link between the solution space $V = \mathrm{Sol}(L)$ and the $k[\partial]$-module M^*. This link is obtained by introducing the map ϕ from V to M^* defined by $\phi(y) = yb_1 + \cdots + y^{(r-1)} b_r$. Calculations with elements of the C-vector space $\mathrm{Sol}(L)$ have their counterparts in the $k[\partial]$-module M^*. For example, one recovers the determinants of the previous sections from the following identity for exterior products in the module $\Lambda^2 M^*$

$$\phi(y_1) \wedge \phi(y_2) = \sum_{1 \leq i < j \leq r} \omega_{i-1,j-1}\, b_i \wedge b_j \qquad \text{with} \qquad \omega_{i,j} = \begin{vmatrix} y_1^{(i)} & y_2^{(i)} \\ y_1^{(j)} & y_2^{(j)} \end{vmatrix}.$$

Again, constructs at the level of solution spaces translate into constructs at the level of the corresponding $k[\partial]$-modules.

6. Tannakian Definition of the Differential Galois Group

This section is based on my (Chyzak's) study and tentatively reflects what was not presented by the speaker for lack of time. It aims at defining differential Galois groups by the Tannakian viewpoint, as an alternative to Kolchin's more traditional and elementary definition by differential extension fields. Interestingly, some properties are easier to derive by the Tannakian viewpoint, for instance that it is a linear algebraic group (i.e., a subgroup of $\mathrm{GL}_n(C)$ and an algebraic variety). Another consequence is the possibility to rephrase algorithms in such a way that differential Galois theory, in the sense of Kolchin, is only used as a classification tool to prove the correction of the algorithms, while calculations take place at the level of modules in a more efficient way. This presentation is based on a discussion with the speaker, on conference proceedings by Ramis and Martinet [8, Part 2, Chapter 1], and on unpublished notes by Churchill [2, 3]. More direct references may be works by Bertrand, Deligne, and Katz. The Tannakian construction has a natural counterpart in difference Galois theory [11, Section 1.4].

For comparison sake, Kolchin's definition of the differential Galois group of a linear differential operator $L \in k[\partial]$ is as follows. Let C be the subfield of constants of k, n be the order of L, and consider the Picard-Vessiot extensions k' of k associated with L, i.e., the differential field extensions of k that contain an n-dimensional C-vector space of solutions of L and do not enlarge the constant field C. Then the differential Galois group of L is defined as the group G of differential field automorphisms (i.e., field automorphisms that respect the differential structure) of *any* Picard-Vessiot extension k' that additionally respect the action of k on k'. This mimics the classical Galois theory for a polynomial $P \in k[X]$, where one introduces the group of field automorphisms of a suitable extension k' of k which contains all solutions of P and restrict to the identity on k. While the (algebraic) Galois group of a polynomial is a subgroup of a permutation group $\mathcal{S}_n$, the differential Galois group of an operator is a subgroup of the linear group $\mathrm{GL}_n(C)$ for the common field of constants C of k and k'.

For its part, instead of a single extension k' of k, the Tannakian presentation simultaneously considers a whole collection of $k[\partial]$-modules, and introduces the differential Galois group as a group of internal transformations on this collection. Crucially, each transformation has to transform all the modules in a way compatible with the linear maps between the modules. Moreover, each module M is associated with a solution set that can be viewed as the kernel of the derivation on M, and the above-mentioned transformations have to be compatible with taking solutions.

At the heart of the Tannakian construction are k-vector spaces V that are closed under the action of an operator ∇ which extends the action of the derivation on k by the Leibniz rule:

$$\nabla(af) = a\nabla(f) + a'f, \qquad \text{when } a \in k \text{ and } f \in V.$$

This makes V a $k[\partial]$-module with ∂ acting by ∇.

From now on, we restrict to $k[\partial]$-modules that are finite-dimensional k-vector spaces. Fundamental examples are the modules $M = k[\partial]/k[\partial]L$ discussed in the previous section. We also restrict to $k = \mathbb{C}(z)$. An element $h \in \ker\nabla$ is called a horizontal vector. As has been explained when discussing dual modules M^*, horizontal vectors in M^* correspond to solutions $y \in k^r$ of the equation $\Delta y = 0$ where $\Delta = d/dz - A$ for $(\nabla b_1, \dots, \nabla b_r)^T = -A^T(b_1, \dots, b_r)^T$ once a basis $(b_1, \dots, b_r)$ of M^* has been chosen. Rather than enlarging the space M where we have a solution for L, as is the case in the traditional differential Galois theory, we now enlarge the coefficient field of M^* so as to ensure the existence of a solution to Δ and thus of horizontal vectors for ∇. To this end, consider a non-singular point $a \in \mathbb{C}$ of the operator L, and introduce the field $\mathcal{M}_a$ of germs of meromorphic functions at a, which is isomorphic to the field of convergent Laurent series $\mathbb{C}\{z-a\}[(z-a)^{-1}]$. By Cauchy's theorem, the C-vector space $\ker\Delta$, where Δ is now viewed as acting on $(\mathcal{M}_a)^r$, is of dimension r and supplies with horizontal vectors of ∇ in $\mathcal{M}_a \otimes_{\mathbb{C}(z)} M$. Note that $\ker\Delta$ and $\ker\nabla$ usually have no more structure than that of C-vector spaces.

As an example, let us consider the $\mathbb{C}(z)[\partial]$-module generated by the Bessel function of the first kind $J_0(z)$. It is a two-dimensional $\mathbb{C}(z)$-vector space with basis $(J_0(z), J_1(z))$, and $J_0' = -J_1$. With the above notation,

$$\begin{pmatrix} \nabla J_0 \\ \nabla J_1 \end{pmatrix} = \begin{pmatrix} J_0' \\ -J_0'' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1/z \end{pmatrix} \begin{pmatrix} J_0 \\ J_1 \end{pmatrix}, \qquad \text{so that} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 1/z \end{pmatrix}.$$

As a result of a simple computation, $h = f_1 J_0 + f_2 J_1$ is a horizontal vector of ∇ if and only if

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker\Delta = \mathbb{C}z \begin{pmatrix} -J_1(z) \\ J_0(z) \end{pmatrix} \oplus \mathbb{C}z \begin{pmatrix} -Y_1(z) \\ Y_0(z) \end{pmatrix},$$

where $Y_\nu(z)$ are the Bessel functions of the second kind, and where J_ν and Y_ν now denote germs of the corresponding functions (their local expansions at $a \neq 0$). This simplifies to $h \in \mathbb{C}z(Y_0 J_1 - J_0 Y_1)$, whence h is a constant by the Wronskian relation $Y_0 J_1 - J_0 Y_1 = 2/\pi z$.

To the module M above and any non-singular point $a \in \mathbb{C}$, we have just associated the $\mathbb{C}$-vector space of horizontal vectors of ∇ in the form of local expansions at a. Denote $\eta_a(M)$ this vector space. We now proceed to associate horizontal vectors to more involved module constructions. Denote $\{M\}$ the smallest class of $k[\partial]$-modules containing M and closed under finite direct sums and products, finite symmetric and exterior products, dualization, and taking the module of homomorphisms between two modules, and submodules. One can extend ∇ from M to any $V \in \{M\}$ in a canonical way; in particular:

$$\nabla|_{V\oplus W} = \begin{pmatrix} \nabla|_V & 0 \\ 0 & \nabla|_W \end{pmatrix}, \qquad \nabla|_{V\otimes W} = \nabla|_V \otimes 1|_W + 1|_V \otimes \nabla|_W.$$

This class becomes a category for the usual $k[\partial]$-module morphisms. The map η_a extends to $\{M\}$ by $\eta_a(V) = \ker(\nabla|_{\mathcal{M}_a \otimes_{\mathbb{C}(z)} V})$. In particular, $\eta_a(V \oplus W) = \eta_a(V) \oplus \eta_a(W)$ and $\eta_a(V \otimes W) = \eta_a(V) \otimes \eta_a(W)$. The crucial fact is that η_a is compatible with the maps that are natural between modules on the one hand and horizontal vector spaces on the other hand. Specifically, any $k[\partial]$-module homomorphism h between two modules V and W induces a C-linear homomorphism $\eta_a(h)$

between $\eta_a(V)$ and $\eta_a(W)$. This makes η_a a functor, in the sense that the diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & W \\ \downarrow{\scriptstyle \eta_a} & & \downarrow{\scriptstyle \eta_a} \\ \eta_a(V) & \xrightarrow{\eta_a(h)} & \eta_a(W) \end{array}$$

is commutative for any two modules V and W.

To relate horizontal vectors at two non-singular points a and b, consider the maps σ that associate with any $k[\partial]$-module V a $\mathbb{C}$-linear map $\sigma(V) : \eta_a(V) \to \eta_b(V)$, subject to the constraint that

$$\begin{array}{ccc} \eta_a(V) & \xrightarrow{\eta_a(h)} & \eta_a(W) \\ \downarrow{\scriptstyle \sigma(V)} & & \downarrow{\scriptstyle \sigma(W)} \\ \eta_b(V) & \xrightarrow{\eta_b(h)} & \eta_b(W) \end{array}$$

is a commutative diagram for any homomorphism $h : V \to W$.

Such a map σ (from $\{M\}$ to the linear morphisms in the category of C-vector space) is called a morphism from (the functor) η_a to (the functor) η_b. The collection of such morphisms when a and b vary is a semigroup for composition. The corresponding notion of isomorphisms (of functors) is obtained when each of the linear maps $\sigma(V)$ is invertible. Two cases are of interest: when $a \neq b$, one of those isomorphisms is provided by analytic continuation along a path from a to b; when $a = b$, the isomorphisms from η_a into itself form a group (the group of automorphisms of the functor η_a). This group is the differential Galois group of L, following the Deligne-Katz definition.

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