Limit Shape Theorems for Partitions

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March 8, 1999

[summary by Sylvie Corteel]

Abstract

Many combinatorial and geometrical problems can be reduced to a problem about partitions of natural numbers or vectors, etc. The main asymptotic question is the behaviour of the shape of such a partition when the statistics or dynamics are fixed. This leads us to the problem of limit shapes. Example: what is the typical limit shape of the uniformly distributed partition of the integers? An explicit answer can be given.

A partition of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 1$, $n(\lambda) = \sum_{i=1}^{N} \lambda_i = n$. The λ_i's are the summands of the partitions. Let $\mathcal{P}_n$ denote the set of all partitions of the integer n and $\mathcal{Q}_n$ the set of partitions of the integer n with distinct summands. Let $r_k(\lambda)$ be the multiplicity of the summand k, that is $r_k(\lambda) = \#\{j \mid \lambda_j = k\}$. Clearly, $n(\lambda) = \sum_k k r_k(\lambda)$ and $N(\lambda) = \sum_k r_k(\lambda)$. Recall that $\#\mathcal{P}_n = p(n)$ is the Euler function and the generating function $\sum_n p(n)x^n$ is $\prod_{i\geq 1}(1-q^i)^{-1}$. The author associates a function φ_λ on $[0,\infty)$ with the partition $\lambda \in \mathcal{P}_n$ by the following rule:

$$\varphi_\lambda(t) = \sum_{k \geq t} r_k(\lambda)$$

φ_λ is a step function, continuous on the right and $\int_0^\infty \varphi_\lambda(t) = n$. Let $a = \{a_n\}_{n\geq 0}$ with $a_n > 0$ for all n; the function

$$\tilde{\varphi}_\lambda(t) = \frac{a_n}{n} \sum_{k \geq a_n t} r_k(\lambda) = \frac{a_n}{n} \varphi_\lambda(a_n t)$$

is φ_λ normed by a_n, so that $\int_0^\infty \tilde{\varphi}_\lambda(t) = 1$.

Let μ^n be the uniform measure on the set $\mathcal{P}_n$ of all partitions of the integer $n : \mu^n(\lambda) = p(n)^{-1}$, $\lambda \in \mathcal{P}_n$; the question is whether one can normalize the partitions in such a way that, in some properly chosen space, the measures μ^n have a weak limit on generalized diagrams, and whether this limit is singular. In the last case, the limit measure is concentrated on a limit shape. An affirmative answer to these questions, as well as explicit formulas for limit shapes, are given in the sequel.

Theorem 1. *The scaling $a = \{a_n\}$ for the uniform measure on $\mathcal{P}_n$ such that a non trivial limit exists in the space of generalized diagrams is $a_n = \sqrt{n}$.*

The same scaling is appropriate for the uniform measure on $\mathcal{Q}_n$.

Theorem 2. *Under the previous scaling, the measures μ^n have a weak limit. This limit is singular and concentrated on a continuous curve.*

The limit shape of the uniformly distributed ordinary partitions and partitions into distinct summands can now be stated.

Theorem 3. *For any $\epsilon > 0$, $0 < x, y < \infty$, there exists n_0 such that for all $n > n_0$*

$$\mu^n\{\lambda \in \mathcal{P}_n \mid \sup_{t\in[x,y]} |\tilde{\varphi}_\lambda(t) - C(t)| < \epsilon\} > 1 - \epsilon,$$

where $C(t) = -(\sqrt{6}/\pi)\ln(1 - e^{\pi t/\sqrt{6}})$, or in more symmetric form

$$e^{-\pi x/\sqrt{6}} + e^{-\pi y/\sqrt{6}} = 1.$$

Theorem 4. *For any $\epsilon > 0$, $0 < x, y < \infty$, there exists n_0 such that for all $n > n_0$*

$$\mu^n\{\lambda \in \mathcal{Q}_n \mid \sup_{t\in[x,y]} |\tilde{\varphi}_\lambda(t) - C(t)| < \epsilon\} > 1 - \epsilon,$$

where $C(t) = -(\sqrt{12}/\pi)\ln(1 + e^{-\pi t/\sqrt{12}})$, or in more symmetric form

$$e^{-\pi y/\sqrt{12}} - e^{-\pi x/\sqrt{12}} = 1.$$

The limit shape can also be obtained for the uniform measure on partitions included in a rectangle, partitions with a given number of summands, vector partitions, ... and other kinds of measures called *multiplicative measures.* The detailed results and links with statistical mechanics are presented in [2].

Bibliography

[1] Andrews (George E.). – *The theory of partitions.* – Addison-Wesley Publishing Co., Reading, Mass., 1976, *Encyclopedia of Mathematics and its Applications*, vol. 2, xiv+255p.

[2] Vershik (A. M.). – Statistical mechanics of combinatorial partitions, and their limit configurations. *Rossiĭskaya Akademiya Nauk. Funktsional'nyĭ Analiz i ego Prilozheniya*, vol. 30, n° 2, 1996.