An Intermediate Value Property for First-Order Differential Polynomials

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[summary by Philippe Dumas & Bruno Salvy]

A theorem of Rubel [4] shows that solutions of algebraic differential equations can present pathological asymptotic behaviours. Therefore when studying differential equations from an asymptotic viewpoint it is natural to restrict to solutions obeying extra smoothness conditions. A convenient context for these questions is provided by Hardy fields, which are defined below. A typical differential equation dealt with by the techniques of this work is

$$F(x,y,y') = (x+y^2)y'^3e^x + yy'\log x + y^4 - e^{e^x} = 0.$$

It is easily seen that for $y = e^{e^x}$, $F(x,y,y')$ is asymptotically positive, while for $y = x$, $F(x,y,y')$ is asymptotically negative. It is therefore natural to wonder whether there exists a solution to this equation whose growth is between that of x and of e^{e^x}. The work summarized here [5] gives a positive answer to this question, and proves that there exists such a solution in a Hardy field.

1. Hardy Fields

A *Hardy field* is a field closed under differentiation, whose elements are germs at ∞ of real-valued functions [1]. (Think of it as the set of possible asymptotic behaviours.) Examples of Hardy fields are the field $\mathbb{R}$ of (germs of) constant functions, the field $\mathbb{R}(x)$ of (germs of) rational functions over $\mathbb{R}$. Hardy fields are named after G. H. Hardy, who proved in [2] that exp-log functions (i.e., functions obtained from $\mathbb{R}(x)$ by field operations, the functions exp and $\log|.|$) form a Hardy field.

The main constraint here is that non-zero elements of Hardy fields have to be invertible, and thus cannot have arbitrarily large zeros. Consequently, since their derivatives belong to the field, they have to be ultimately monotonic and tend to a possibly infinite limit. Also, differences of two (germs of) functions of a Hardy field are also in the field and possess a limit, so that this field is ordered. A Hardy field $\mathbb{K}$ can be extended by a C^∞ function y if for all polynomials $P \in \mathbb{K}[Y]$, $P(y)$ is either 0 or does not have arbitrarily large zeros. A Hardy field $\mathbb{K}$ can be extended by real solutions of polynomials in $\mathbb{K}[Y]$ and by antiderivatives of elements of $\mathbb{K}$ [3]. This is how exp-log functions can be built from $\mathbb{R}(x)$.

The order induces a natural topology, a basis of the open sets being given by the open intervals. Thus continuous functions are defined and for instance, the differentiation operator is continuous since $y' > f$ implies $y > \int f$. This is an open set since the field can be extended by $\int f$ if necessary.

The aim of this work is to prove the following.

Theorem 1. *Let K be a Hardy field and $F \in \mathbb{K}[x,y]$. Assume there exist ϕ and ψ in $\mathbb{K}$ such that $F(\phi,\phi') < 0 < F(\psi,\psi')$. Then there exists η in a Hardy field extension of $\mathbb{K}$ such that $F(\eta,\eta') = 0$.*

The function $y \mapsto F(y,y')$ is continuous, but in general Hardy fields are not Archimedean (consider 1 and x in $\mathbb{R}(x)$). Consequently, the intermediate value theorem may not hold. The proof consists in lifting properties of continuous functions over $\mathbb{R}$ to Hardy fields. The same question for higher-order differential polynomials is still a conjecture.

2. Basic Case

We first consider equations of the form

$$y'(x) = G(x, y(x)), \tag{1}$$

where G is C^1 in the neighborhood of $\mathcal{S} = \{(x,y), r \leq x, a(x) \leq y \leq b(x)\}$, for some $r \in \mathbb{R}$ and a and b in a Hardy field $\mathbb{K}$ giving different signs to $y'(x) - G(x, y(x))$. A simple reasoning based on the intermediate value theorem shows the existence of a C^1 solution $\eta(x)$ of (1) with $a(x) < \eta(x) < b(x)$ for $x \geq r$.

If, moreover, $x \mapsto G(x, h(x))$ belongs to $\mathbb{K}$ for all h in $\mathbb{K}$, then η belongs to an extension of $\mathbb{K}$. This is proved in three steps. First, if η has arbitrarily large zeros then so does $x \mapsto G(x, 0)$ but since this belongs to a Hardy field, we get $G(x, 0) = 0$ and $\eta = 0$ is the unique solution of the differential equation. Next, if $h \neq \eta$ belongs to an extension of $\mathbb{K}$, then $\theta = \eta - h$ cannot have arbitrarily large zeros, using the same argument as before with the equation

$$\theta'(x) = G(x, \theta(x) + h(x)) - h'(x).$$

Noting that any polynomial $P \in \mathbb{K}[Y]$ can be factored in linear factors or quadratic factors with negative discriminant extends this argument to $P(\eta)$ and concludes the proof.

3. General First Order Case

The aim is to reduce the general case $F(y, y') = 0$ to the basic case considered in the previous section. This is achieved through an analogue of the cylindric-algebraic decomposition: the interval (ϕ, ψ) is split into subintervals $\phi = a_1 < \cdots < a_n = \psi$ such that in every interval (a_i, a_{i+1}) there are finitely many functions $f_{i,j}$ algebraic over $\mathbb{K}$ and the polynomial $F(y, z)$ has constant sign in the cell $a_i < y < a_{i+1}$, $f_{i,j}(y) < z < f_{i,j+1}(y)$. Note that everything here also depends on x through the coefficients of F.

It is now sufficient to exhibit two functions a, b, with $a_i < a < b < a_{i+1}$ for some i, such that $a' - f_{i,j}(a) < 0 < b' - f_{i,j}(b)$ for some j. Let A be the set of $y \in (\phi, \psi)$ such that $F(y, y') < 0$ and similarly B for $F(y, y') > 0$. If A (resp. B) has an upper (resp. lower) bound, then this is a solution of the equation and we are done. Otherwise, it is possible to select $a \in A$ and $b \in B$ belonging to the same interval (a_i, a_{i+1}) (if not, one of the a_i's would be a solution of the equation). Necessarily, (a, a') and (b, b') do not belong to the same cell and therefore one of the $f_{i,j}$ fulfills our needs. We denote it f. The reduction to the basic case requires that the application $(x, y) \mapsto f(y)$ be C^1 in the domain $\mathcal{S}$. This follows from the analyticity of the roots of a polynomial equation with respect to its coefficients outside of singular varieties and the C^1 property of these coefficients for x sufficiently large.

Bibliography

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