The Local Limit Theorem for Random Walks on Free Groups

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[summary by Cyril Banderier]

Abstract

Local limit theorems and saddlepoint approximations are given for random walks on a free group whose step distributions have finite support. These are derived by exploiting a set of algebraic relations among certain generating functions that arise naturally in connection with the transition probabilities of the random walks. Basic tools involved in the analysis are the elementary theory of algebraic functions, the Perron-Frobenius theory of nonnegative matrices, and standard techniques of singularity analysis.

1. Walks on Groups

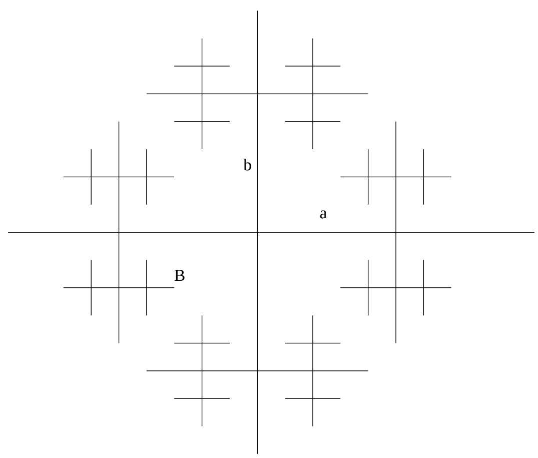

FIGURE 1. A representation (Cayley graph) of the infinite free group $\mathbb{Z} \star \mathbb{Z}$

Let $\mathbb{G}$ be a the free group with generators $a_1, \dots, a_L$. For example, the free group $\mathbb{Z} \star \mathbb{Z}$ has two generators a and b. The word $aba\bar{a}\bar{b}\bar{a}\bar{a}\bar{b}aa$ corresponds to the point B, the associated reduced word is $\bar{a}\bar{b}aa$.

A finite-range random walk $\{Z_n\}_{n\geq 0}$ is a Markov chain on $\mathbb{G}$ with $Z_0 := e$ (the identity of the group, the "origin", the starting point) and transition probabilities

$$\Pr\{Z_{n+1} = yx | Z_n = y\} = p_x \qquad \forall x, y \in \mathbb{G}, n \geq 0,$$

where p_x for $x \in \mathbb{G}$ is a probability distribution with finite support (in other words, p_x is the probability of the "jump" x).

Note $p^{*n}(x)$ the probability of being in x after n steps. It is assumed that the random walk is irreducible and aperiodic, that is, that

$$\forall x \in \mathbb{G} \qquad \sum_{n \geq 1} p^{*n}(x) > 0 \qquad \text{(irreducibility);}$$
$$\mathrm{GCD}\{n; p^{*n}(e) > 0\} = 1 \qquad \text{(aperiodicity).}$$

Another important condition is the following:

Positivity: $\quad p_e > 0$ and $p_g > 0$ for all generators g of $\mathbb{G}$ (and their inverses).

Similarly to the random walk in the Euclidian case $\mathbb{Z}^d$, a local limit theorem is given by the asymptotics

$$p^{*n}(x) \sim \frac{B_x R^{-n}}{\sqrt{2\pi R} n^{3/2}}.$$

Similar results were already known when all the steps are of size one (*nearest neighbour* random walk [3]) or when all the words at a same distance from the origin are equiprobable (the so-called *isotropic* random walk [2, 8, 9]).

2. **Singularity Analysis**

This section is devoted to the analysis of some probability generating functions (PGF) related to the walk. For $x \in \mathbb{G}$ and $z \in \mathbb{C}$ ($|z| < 1$), define

- the random variable coding where one is after n steps: Z_n,
- the PGF to reach x in n steps: $G_x(z) := \sum_n p^{*n}(x) z^n$,
- the PGF of the excursions (Green's function): $G(z) := G_e(z)$,
- the first time x is reached: $T_x := \inf\{n \geq 0 : Z_n = x\}$,
- the PGF to reach x for the first time in n steps: $F_x(z) := \sum_n \Pr\{T_x = n\} z^n$.

Note that aperiodicity and irreducibility imply that for all sufficiently large $n \geq 1$, $p^{*n}(e) > 0$ and $p^{*n}(y) > 0$, for any (inverse of a) generator y.

The following (combinatorially trivial) relations

$$G_x(z) = F_x(z) G(x),$$

$$G(z) = 1 + z \left(p_e + \sum_{x \neq e} p_x F_{x^{-1}}(z) \right) G(z) = \left(1 - z p_e - z \sum_{x \neq e} p_x F_{x^{-1}}(z) \right)^{-1}$$

allow to prove that all of the functions F_x and G_x have the same radius of convergence R, $1 < R < \infty$ (the less obvious is that R is *strictly* greater than 1).

Let $\mathbb{B}$ be the set of points at distance $\leq K$, where K is such that there is no smaller ball in which the support of the step distribution $\{p_x\}$ is contained. Define now

- the first time that x is exceeded: τ_x
- the PGF to go from a to xb while x as never been exceeded before:

$$H_x^{ab} = \sum_n \Pr\{Z_n = xb | Z_0 = a \text{ and } Z_{i<n} \notin x\mathbb{B}\} z^n$$

- the PGF to go from x to the origin: $F_{x^{-1}}(z)$

The formula

$$\forall x \neq e \qquad F_x(z) = \sum_{b \in \mathbb{B} - \{e\}} H_x^{eb}(z) F_{b^{-1}}(z)$$

leads to an expression of $F_x = uH_x(z)v = uH_{x_1}(z) \cdots H_{x_m}(z)v$ where u is the projection on e and v a vector whose entries are the $F_{b^{-1}}(z)$ for $b \in \mathbb{B}$ and where the product of matrices is over $x_1 \cdots x_m$, the reduced word associated to x.

It is then proven by the Markov property that all the non-constant $H_{x_i}^{ab}$ satisfy polynomial relations ($H_{x_i} = Q_i(H_{x_1}, H_{x_2}, \dots)$, see [6] for exact relations) and that they have the same radius of convergence.

By elimination (Gröbner basis or resultants), the functions F_x and G_x are algebraic. Their Puiseux expansion leads to an algebraic singularity with exponent $1/2$.

Here is a sketch of the proof that the exponent is indeed $\alpha = 1/2$. Define J_z the Jacobian matrix $\left(\partial Q_i / \partial H_{x_j}\right)$, the polynomials Q_i have nonnegative coefficients, thus there exists n such that J_z^n is an aperiodic and irreducible matrix with strictly positive coefficients. By the Perron-Frobenius theorem, J_z has a positive eigenvalue λ_z of multiplicity 1. The function λ_z is increasing and real-analytic and $\lambda_R = 1/R$. Considering a left eigenvector of J_R and using the shape of the Q_i yields to the relations $(R-z)(C + \cdots) = C'(R-z)^{2\alpha} + \cdots$, thus $2\alpha = 1$.

As $z = R$ is the dominant singularity of F_x and G_x, one has the two following theorems:

Theorem 1 (Local limit theorem, access). *Assuming irreducibility and aperiodicity, one has, for a positive constant B_x:*

$$p^{*n}(x) \sim \frac{B_x}{\sqrt{2\pi R} R^n n^{3/2}}.$$

Theorem 2 (Local limit theorem, first access). *Assuming positivity, one has, for a positive constant A_x:*

$$\Pr\{x \textit{ is reached for the first time after } n \textit{ steps}\} \sim \frac{A_x}{\sqrt{2\pi R} R^n n^{3/2}}.$$

3. Saddlepoint Approximations

The probability to reach a point x at a distance m of the origin in n steps is

$$p^{*n}(x) \sim \frac{\exp(n\beta(m/n))}{\sqrt{\ddot{\psi}(m/n)}} C(m/n)$$

for appropriate functions β, C and $\ddot{\psi}$ (see the correct definitions / notations in [6]).

This uniform asymptotics in x and n corresponds to the classical saddlepoint approximation (sharp large deviations theorems) for sums of iid random vectors in $\mathbb{R}^d$.

The saddlepoint approximations are of interest for another reason. For large n, nearly all the mass in the probability distribution $p^{*n}(x)$ is concentrated in the region $|x| \geq \epsilon n$, where the local limit approximations are not accurate. This contrasts with the situation for finite range random walk in Euclidean space. In fact, Guivarch [4] has shown that for random walks in $\mathbb{G}$, the distance from the origin grows linearly in n. Sawyer and Steger [10] have further shown that $(|Z_n| - n\beta)/\sqrt{n}$ converges in law to a normal distribution.

Finally, S. Lalley, using a special matrix product and results on Ruelle's Perron-Frobenius operators, derives a saddlepoint approximation, uniformly for m/n in a given compact

$$\Pr\{|Z_n| = m\} \sim \frac{\exp(nB(m/n))}{\sqrt{2\pi m D(m/n)}} C(m/n).$$

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