Loop-Erased Random Walks

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[summary by Cyril Banderier]

Abstract

The loop-erased random walk is the simple curve obtained by removing in the chronological order the loops of the original random walk. A basic aspect of these walks in $\mathbb{Z}^2$ is studied: its average length (thus solving a conjecture of Guttmann). The techniques are combinatorial and use a bijection due to Temperley between maximal trees and perfect coupling.

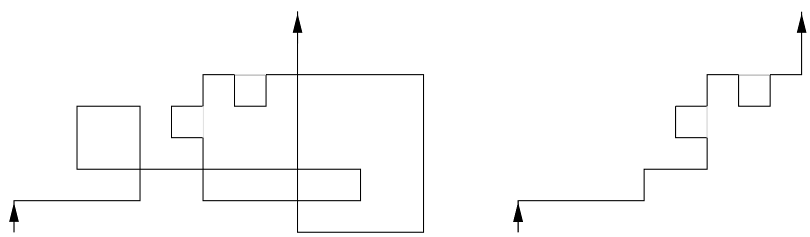


FIGURE 1. A random walk and its associated LERW

1. LERW=ST=DT

This is not a new equality between new complexity classes but it simply emphasizes the fact that loop-erased random walk (LERW), spanning tree (ST) and domino tiling (DT) are essentially the same object.

Indeed, in [12] shows that, for a uniformly chosen spanning tree on a region of $\mathbb{Z}^2$, the unique arc (branch) between two points has the same distribution as the LERW between these two points. Moreover, Temperley [13] gives a constructive one-to-one correspondence between spanning trees and domino tilings.

From the other talk of Richard Kenyon, we know that domino tiling (the so-called two dimensional lattice dimer model, a model which has some ties with Ising model) is the only nontypical *ad hoc* statistical physical model where conformal invariance is proved, so representations of the Virasoro algebra [15] could help finding critical exponents.

In order to solve this "self-avoiding walk model" (*i.e.*, to set the critical exponent), R. Kenyon does not use representation theory, but a "discrete Laplacian", from which he gets the full asymptotics. Whereas a lot is known about properties of the continuous Laplacian [5], works on the discrete Laplacian are more recent [10].

As conjectured by Bursill and Guttmann [4], the exponent for LERW is 5/4. Richard Kenyon proves this by applying the "equality" LEWR=ST on the following theorem

Theorem 1. *On the uniform spanning tree process on $\mathbb{N}\times\mathbb{Z}$, the expected number of vertices on the arc from $(0,0)$ to ∞ which lie within distance N of the origin is $N^{5/4+o(1)}$.*

The next theorem is sometimes attributed to Kirschoff [11] and proven in [1].

Theorem 2 (Matrix-tree Theorem). *For a graph G with set of vertices $\{v_i\}$, let*

$$\Delta := \begin{pmatrix} \deg(v_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \deg(v_n) \end{pmatrix} - \mathrm{Adj}(G),$$

then the number of spanning trees of G is the product of the nonzero eigenvalues of Δ divided by the size of G. For an $m\times n$ rectangle, the number of spanning trees is thus

$$\prod_{\substack{(j,k)\neq 0 \\ j=0,\dots,m-1 \\ k=0,\dots,n-1}} \left(4 - 2\cos(\frac{\pi k}{n}) - 2\cos(\frac{\pi j}{m})\right).$$

It is possible to extend this kind of result to a class of polygons which are decomposable in rectangles, the so-called Temperleyan polyominoes. (Triangulations are also a way, as there is a determinant-like expression for triangles.)

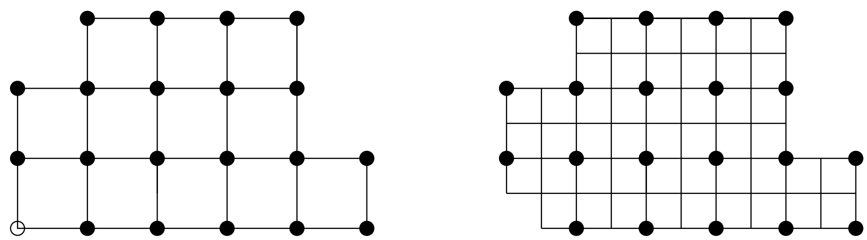


FIGURE 2. A graph and its associated Temperleyan polyomino

The number of spanning trees of the graph is the number of domino tilings of its Temperleyan polyomino. Taking the log in the previous formula gives a special case of the following theorem:

Theorem 3. *Let $U\subset\mathbb{R}^2$ be a rectilinear polygon with V vertices. For each $\epsilon>0$, let P_ϵ be a Temperleyan polyomino in $\epsilon\mathbb{Z}^2$ approximating U in the natural sense (the corners of p_ϵ are converging to the corners of U). Let A_ϵ be the area and* Perim_ϵ *be the perimeter of P_ϵ. Then the log of the number of domino tilings of P_ϵ is*

$$\frac{c_0 A_\epsilon}{\epsilon^2} + \frac{c_1\,\mathrm{Perim}_\epsilon}{\epsilon} + O(\log(\epsilon))$$

where $c_0=\frac{G}{\pi}$, $G = 1-\frac{1}{3^2}+\frac{1}{5^2}-\cdots$ is Catalan's constant, $c_1=\frac{G}{2\pi}+\frac{\log(\sqrt{2}-1)}{4}$.

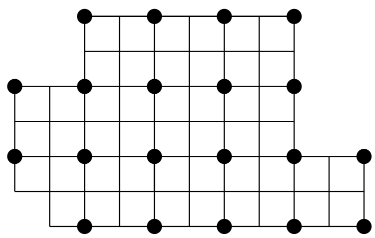
According to the author, "Part of the motivation for the above theorem is to validate a certain heuristic, which attempts to explain how the presence of the boundary affects the long-range structure of a random tiling. In particular it attempts to explain how the boundary affects the densities of local configurations far from the boundary [3]. This heuristic is called the 'phason strain' principle. The heuristic is as follows: The boundary causes the *average height function* of a tiling to deviate slightly from its entropy-maximizing value of 0. At a point in the region where the average height function has nonzero slope, the "local" entropy there is smaller than the maximal possible entropy, by an amount proportional to the square of the gradient of the average height function.

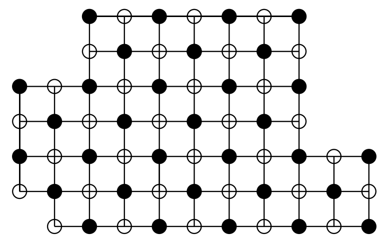
The system behaves in such a way as to maximise the total entropy subject to the given boundary values of the height function, and the resulting average height function is the function which minimises the (integral of) the square gradient. That is, the average height function is harmonic."

Anyway an exact application of the phason strain principle gives in fact slightly different asymptotics (from the one given in the theorem 3), so this principle is not totally valid here, but however it gives a good approximation.

2. **Height Function**

For a given tiling, the height function h is easily defined [14] by bicolouring the Temperleyan polyomino (there are no adjacent vertices of the same colour):





then, for each oriented edge AB on the border of a domino,

$$h(A) - h(B) := \begin{cases} +1 & \text{if the square on the left of } AB \text{ is black,} \\ -1 & \text{otherwise.} \end{cases}$$

Note that, up to an arbitrary additive constant, the height of the boundary is independent of the tiling.

For a smaller and smaller lattice (e.g., $\epsilon\mathbb{Z}^2$), one can approximate any domain U of $\mathbb{C}$. For a very fine lattice (in fact, taking the limit when $\epsilon \to 0$), one can study the probability of appearance in the tiling of some patterns, their repartition in the tiling and also how the shape of the boundary influences the tiling. It appears that there are links with conformal theory, as explained below.

Let P_ϵ be a Temperleyan polyomino associated to a rectilinear polygon U of $\mathbb{C}$. Now, let h_ϵ be the average height function of P_ϵ, that is the average height over all domino tilings of P_ϵ and then define

$$h(x) := \lim_{\epsilon \to 0} h_\epsilon(x_\epsilon).$$

For $x \in \partial U$, $h(x)$ is defined by continuity from values of h in the interior.

The remarkable fact is that this limiting average height function h can be expressed as

$$h(v) = \frac{4}{\pi} \Im \int_{b_0}^{v} \lim_{z \to v} \left(F_+(v,z) - \frac{2}{\pi(z-v)} \right) du = -\frac{2}{\pi} \Im \log \wp'(z),$$

where $\wp$ is the Weierstrass elliptic function and where $F_+(u,z)\,du$ is a meromorphic 1-form, thus allowing some links with conformal mapping theory. We refer to "Conformal invariance of domino tiling" (1997) and to "The asymptotic determinant of the discrete Laplacian" (1999) for further informations[1].

[1] Like other recent preprints of the author, they are available at his home page
`http://topo.math.u-psud.fr/~kenyon/`

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