Explicit Sufficient Invariants for an Interacting Particle System

Yoshiaki Itoh
Institute of Statistical Mathematics, Japan

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[summary by Philippe Robert]

Abstract

We introduce a new class of interacting particle systems on a graph G. Suppose there are initially $N_i(0)$ particles at each vertex i of G, and that the particles interact to form a Markov chain: at each instant two particles are chosen at random, and if these are at *adjacent* vertices of G, one particle jumps to the other particle's vertex, each with probability $1/2$. The process enters a death state after a finite time when all the particles are in some *independent* subset of the vertices of G, i.e., a set of vertices with no edge between any two of them. The problem is to find the distribution of the death state $\eta_i = N_i(\infty)$ as a function of the numbers $N_i(0)$.

We are able to obtain, for some special graphs, the *limiting* distribution of each N_i if the total number of particles $N \to \infty$ in such a way that the fraction $N_i(0)/N = \xi_i$ at each vertex is held fixed as $N \to \infty$. In particular we can obtain the limit law for the graph $S_2 : \cdot\!\!-\!\!\!-\!\!\cdot\!\!-\!\!\!-\!\!\cdot$ having 3 vertices and 2 edges.

This talk is based on a joint paper with Colin Mallows and Larry Shepp [1]

1. A Particle System

If G is a connected graph, the following particle system is considered. Initially n particles are distributed on the nodes of the graph, at each unit of time two particles are chosen at random if they are on neighboring nodes then one of the two particles jumps to the other one's vertex, each with probability $1/2$. This kind of model has various applications to genetics, to voting and symbolic computation.

It is easily seen that if a vertex has no particles it remains empty and the process lives on a subgraph with this vertex removed. The process continues until all vertices are empty except those of an *independent* subset J of G, i.e., the vertices in J have no edge in common. At that time the process stops since no interaction can occur anymore. We denote by $N(t) = (N_i(t); i \in G)$ the vector of the number of particles on the vertices at time t, τ is the first time the process reaches an independent set; $N(\tau)$ is the terminal state of the process.

Example (The complete graph). The singletons are obviously the independent sets for this graph and by symmetry the terminal distribution is easy to derive, $\Pr(N(\tau) = \delta_i) = N_i(0)/n$.

In general it is difficult to determine the distribution of the final state of the process. The method used here is to find functionals f such that the relation

$$\mathbb{E}(f(N(t))) = \mathbb{E}(f(N(0)))$$

holds for all $t \geq 0$ and for the random time $t = \tau$. If there are sufficiently many functions f then the distribution of $N(\tau)$ may be derived. In a classical dynamical system such a function is an *invariant* of the motion. In a probabilistic context the corresponding notion is the martingale, a function f is *admissible* if $(f(N(t))$ is a martingale, i.e., for all $s \leq t$,

$$\mathbb{E}(f(N(t))|\text{all events before time } s) = f(N(s)),$$

in particular $\mathbb{E}(f(N(t))) = \mathbb{E}(f(N(0)))$ for all $t \geq 0$.

In our case $\left(\sum_{i\in G} N_i(t)\right)$ is a trivial (constant!) martingale. Another example is the process $(N_i(t))$ for $i \in G$, it is also a martingale, hence $\mathbb{E}(N_i(\tau)) = N_i(0)$.

For an admissible f, under mild integrability conditions we shall have $\mathbb{E}(f(N(\tau))) = f(N(0))$, hence if there is a rich class of such f the distribution of $N(\tau)$ will be determined. It turns out that this discrete model does not seem to have sufficiently many admissible functions. The situation is somewhat simpler if one considers a continuous version $(X_i(t); i \in G)$ of the process, it is the solution of the stochastic differential equation

$$dX_i = \sum_{j\in\mathcal{N}_i} \sqrt{X_i X_j}\, dB_{ij}, \tag{1}$$

where $\mathcal{N}_i$ is the set of the neighbors of $i \in G$. The B_{ij}, $i, j \in G$ are Brownian motions such that $B_{ij} = -B_{ji}$, so that if

$$\sum_{i\in G} X_i(t) = 1$$

for $t = 0$ then this relation holds for all $t \geq 0$ (the "number" of particles is constant). Notice that if one of the coordinates is 0 it remains 0 as in the discrete model.

2. Martingales of the Continuous Process

A star graph S_r with $r+1$ vertices is considered, the vertex 0 is supposed to be the center.

Proposition 1. *If $G = S_r$ and $\alpha_1 + \cdots + \alpha_r = 0$, the process $(P_n^\alpha(X_1(t), \ldots, X_r(t)))$ is a martingale, where P_n^α is the polynomial defined by*

$$P_n^\alpha(x_1, \ldots, x_r) = \sum_{\substack{i_1\geq 1,\ldots,i_r\geq 1\\ i_1+\cdots+i_r=n}} \binom{n}{i}\binom{n-r}{i-1}\prod_{k=1}^{r}(\alpha_k x_k)^{i_k},$$

with $i = (i_1, \ldots, i_r)$ and $\binom{n}{i} = n!/i_1!\cdots i_r!$.

The proof is carried out with the help of Itô's equation. In this manner there is a sequence of martingales associated to the stochastic process $(X(t))$.

A similar situation occurs with the Brownian motion (B_t), if h_n is the n-th Hermite polynomial, then

$$(M_n(t)) = \left(t^{n/2} h_n\left(B(t)/\sqrt{t}\right)\right)$$

is a martingale. The appropriate generating function of these processes gives the classical exponential martingale

$$\sum_{n=0}^{+\infty} \frac{c^n}{n!} M_n(t) = \exp\left(cB(t) - \frac{c^2}{2}t\right).$$

The martingales $(M_n(t))$ are not easy to use to get distributions of the Brownian motion functionals. The above exponential martingale is simple, "contains" all the martingales $(M_n(t))$ and many distributions related to the Brownian motion can be directly obtained with it (see Rogers and Williams [3]). In the following a similar method is used to get the terminal distribution of the process. As we shall see the corresponding exponential martingale is not as simple as for Brownian motion but it will give the desired distribution.

3. Star Network With Three Nodes

Since the expression of the polynomials P_n^α is rather complicated to derive results on the distribution of $(X(\tau))$, the simpler case $r = 2$ is now considered. In this case the terminal states $(X_0(\tau), X_1(\tau), X_2(\tau))$ are $(1,0,0)$ or $(0, x, 1-x)$, $x \in [0,1]$. Taking $\alpha_1 = -1$ and $\alpha_2 = 2$ in the above proposition, we get that, for $n \geq 2$,

$$(Y_n(t)) = \left(\sum_{i=1}^{n-1} \binom{n}{i}\binom{n-2}{i-1}(-1)^i X_1(t)^i X_2(t)^{n-1}\right) \tag{2}$$

is a martingale. The key result of this section is the following identity, for $|v| < 1/4$ and $0 \leq x \leq 1$

$$\sum_{n\geq 2} \frac{v^n}{n} \sum_{i=1}^{n-1} \binom{n}{i}\binom{n-2}{i-1}(-1)^i x^i (1-x)^{n-1} = xv + \frac{1-v}{2}\left(1 - \sqrt{1 + \frac{4xv}{(1-v)^2}}\right), \tag{3}$$

this suggests to sum the expressions (2) as follows

$$Z_u(t) = \sum_{n\geq 2} \frac{u^n}{n} Y_n(t).$$

The process $(Z_u(t))$ is also a martingale, the exponential martingale of $(X(t))$, and the representation (3) gives the following result. If μ is the terminal distribution of $(X_1(t))$ with the initial state $X_i(0) = \xi_i$, $i = 1, \ldots, r$, then for $|u| < 1/4$,

$$\int_0^1 \sqrt{(1-u)^2 + 4ux}\,\mu(dx) = u(\xi_1 + \xi_2 - 1) + \sqrt{1 + 2u(\xi_1 - \xi_2) + u^2(\xi_1 + \xi_2)^2}.$$

The problem is thus reduced to a kind of moment problem (by differentiating with respect to u under the integral). Further analytic manipulations give the density f of μ as

$$\frac{d^2}{dx^2}\frac{2}{\pi}\int_0^x \frac{1}{\sqrt{x-w}} g(w)\,dw,$$

where g is a complicated function but with an explicit expression.

Bibliography

[1] Itoh (Yoshiaki). – Random collision models in oriented graphs. *Journal of Applied Probability*, vol. 16, n° 1, 1979, pp. 36–44.

[2] Itoh (Yoshiaki), Mallows (Colin), and Shepp (Larry). – Explicit sufficient invariants for an interacting particle system. *Journal of Applied Probability*, vol. 35, n° 3, 1998, pp. 633–641.

[3] Rogers (L. C. G.) and Williams (David). – *Diffusions, Markov processes, and martingales. Vol. 2: Itô calculus.* – John Wiley & Sons Inc., New York, 1987, xiv+475p.