

Colouring Rules for Finite Trees and Probabilities of Monadic Second Order Sentences

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Abstract

Given a set of colouring rules applying to the vertices of any finite rooted tree, we study the asymptotic behaviour of the probability that an n vertex tree has a given root colour. These results will prove that the fraction of labelled or unlabelled rooted trees satisfying any fixed monadic second-order sentence converge to limiting probabilities.

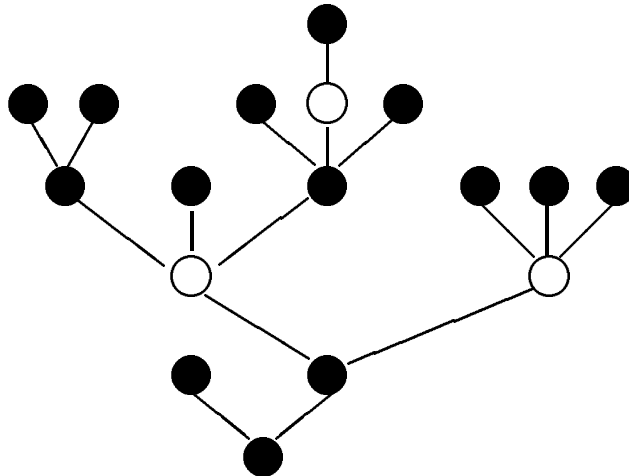
1. Introduction

Given a finite rooted tree and a set of k colours, the vertices are coloured from the leaves to the root according to a set of colouring rules, namely a function $h : \mathbb{N}^k \rightarrow \{1, 2, \dots, k\}$. The colour assigned to a vertex depends only on the number C_1, \dots, C_k of its immediate predecessors having colour $1, \dots, k$.

Example. Let

$$h(C_{black}, C_{white}) = \begin{cases} \text{black} & \text{if } C_{black} \text{ is even} \\ \text{white} & \text{if } C_{black} \text{ is odd} \end{cases}$$

be a set of colouring rules. From the definition of h , the leaves of the following tree are coloured black and we find its root colour is black.



Note that the root of a finite rooted tree is black iff the number of its vertices is odd, with the set of colouring rules defined above.

Let $\mu_n[i]$ be the fraction of n vertex labelled trees with root colour i .

Theorem 1. *Let $\mu[i] = \lim_{n \rightarrow \infty} \mu_n[i]$ and the corresponding Cesàro limit*

$$\bar{\mu}[i] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \mu_m[i].$$

For any set of colouring rules h , $\bar{\mu}[i]$ exists for all colours $i = 1, \dots, k$ and either

1. $\bar{\mu}[i] > 0$ or
2. $\exists c > 1$ such that $\mu_n[i] < c^{-n}$ for all sufficiently large n .

Although the existence of $\mu[i]$ implies the existence of $\bar{\mu}[i]$ in general, the converse needs additional conditions to be true.

2. Applications to Logic

2.1. First Order Logic. There exists an analogous result for first order sentences about a graph. The language in which these sentences are written contains the usual quantifiers, parentheses and connectives with an additional predicate symbol $E(x, y)$ expressing the fact that vertex x and vertex y are joined by an edge.

Example. The following expression is a first order logic sentence expressing “every vertex has degree 2”:

$$\forall x \exists y_1 \exists y_2 (\neg y_1 = y_2 \wedge \forall z (E(x, z) \Leftrightarrow z = y_1 \vee z = y_2)).$$

Fagin [3], Glebskiĭ, Kogan, Liogon’kiĭ and Talanov [4] have proved the following result

Proposition 1. *Let $\mu_n(\varphi)$ be the fraction of n vertex graphs with property φ . For every first order sentence φ about a graph, $\mu(\varphi) = \lim_{n \rightarrow \infty} \mu_n(\varphi)$ exists and $\mu(\varphi) = 0$ or 1.*

That the only possible values are 0, 1 is a consequence of the fact that graphs have no roots.

2.2. Monadic Second Order Logic. The situation for monadic second order sentences about a rooted tree is quite different since the language provides a constant symbol R denoting the root, and it can handle sets of vertices using second order variables.

Determining the satisfiability of a monadic second order sentence φ of rank r reduces to finding the root colour of a rooted tree \mathcal{T} for a particular system of colouring rules. Results arising from Compton’s method of components [2] establish that if φ is a sentence of rank r , then there exists sentences ψ_1, \dots, ψ_k of rank r such that:

1. Every finite rooted tree satisfies exactly one ψ_i ;
2. Every φ of rank r is equivalent to $\bigvee_{i \in S} \psi_i$ for some set S .

If \mathcal{T} is a rooted tree that has component trees $\mathcal{T}_1, \dots, \mathcal{T}_m$ that satisfy sentences $\psi_{i_1}, \dots, \psi_{i_m}$, then there exists a unique i such that \mathcal{T} satisfies ψ_i , and this particular i can be interpreted as the root colour of \mathcal{T} . (For details see [8]).

2.3. Boolean Formulas. Assume we have M boolean variables x_1, \dots, x_M . Then the colours turn out to be the 2^{2^M} boolean functions Ψ_i . The existence of the limiting probability $\mu[i]$ is stated in the following theorem:

Theorem 2. *Let $\mu_n[i]$ be the fraction of formulas of size n which compute the boolean function Ψ_i , $i \in \{1, \dots, 2^{2^M}\}$. Then $\lim_{n \rightarrow \infty} \mu_n[i] = \mu[i]$ exists and $\mu[i] > 0$.*

3. Enumeration of Rooted Trees

3.1. Labelled Rooted Trees. We use generating functions methods to determine $\bar{\mu}[i]$ in the labelled case. Note that a similar proof can be done for the unlabelled case.

Let $T(x)$ denote the generating function for labelled rooted trees:

$$T(x) = t_1x + \frac{t_2}{2!}x^2 + \frac{t_3}{3!}x^3 + \cdots + \frac{t_n}{n!}x^n + \cdots$$

where t_i is the number of i vertex labelled rooted trees. Since this structure is decomposable, we easily obtain a functional equation on $T(x)$ and find:

$$T(x) = xe^{T(x)}.$$

Hence, using Lagrange inversion we get:

$$T(x) = x + \frac{2}{2!}x^2 + \frac{3^2}{3!}x^3 + \cdots + \frac{n^{n-1}}{n!}x^n + \cdots.$$

The radius of convergence of this series is $\rho = 1/e$, $x = \rho$ is the only singularity on the circle of convergence, where there exists a constant h_1 such that $T(x)$ behaves like $1 + h_1\sqrt{\rho - x}$. One can then apply Darboux's theorem and find that t_n behaves asymptotically like $t_n \sim C\rho^{-n}n^{-3/2}$.

3.2. Labelled Trees with a Particular Root Colour. Let $T_i(x)$ be the generating function for labelled trees with root colour i ,

$$T_i(x) = x \sum_{\substack{M_1, \dots, M_k \\ h(M_1, \dots, M_k) = i}} \frac{T_1^{M_1}(x)}{M_1!} \cdots \frac{T_k^{M_k}(x)}{M_k!}.$$

To find $y_i = T_i(x)$ we have to solve the system:

$$\{y_i = g_i(x, y_1, \dots, y_k)\}_{i \in \{1, \dots, k\}} \text{ where } g_i(x, y_1, \dots, y_k) = x \sum_{\substack{M_1, \dots, M_k \\ h(M_1, \dots, M_k) = i}} \frac{y_1^{M_1}}{M_1!} \cdots \frac{y_k^{M_k}}{M_k!}.$$

4. Cesàro Probabilities

To determine probability $\bar{\mu}[i]$ we use a partial converse of the following Abelian theorem:

Theorem 3. Let $b(x) = \sum_{n \geq 0} b_n x^n$, $c(x) = \sum_{n \geq 0} c_n x^n$ and ρ be the radius of convergence of $b(x)$. If $\lim_{n \rightarrow \infty} c_n/b_n = \mu$ and $\sum_{n \geq 0} b_n \rho^n$ diverges then:

$$\lim_{x \rightarrow \rho^-} c(x)/b(x) = \mu.$$

Setting $c(x) = T'_i(x)$ and $b(x) = T'(x)$, we find that the conditions above are satisfied since $\lim_{x \rightarrow \rho^-} T'(x) = \infty$. The result is given by the following Tauberian theorem:

Theorem 4 (Compton [1]). Let $b_n \sim Cn^\alpha$, $\alpha > -1$, $b_n \neq 0$ for $n \geq 0$, $c_n = O(b_n)$. If

$$\lim_{x \rightarrow \rho^-} c(x)/b(x) = \mu,$$

where ρ is the radius of convergence of $b(x) = \sum_{n \geq 0} b_n x^n$ then:

$$\bar{\mu} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{M=1}^n \frac{c_M}{b_M} = \mu.$$

Since

$$T_i'(x) = \frac{\partial g_i}{\partial x}(x, T_i(x), \dots, T_k(x)) + \sum_{j=1}^k T_j'(x) \frac{\partial g_i}{\partial y_j}(x, T_i(x), \dots, T_k(x)),$$

when $x \rightarrow \rho-$, we have

$$\bar{\mu}[i] = \sum_{j=1}^k \frac{\partial g_i}{\partial y_j} \bar{\mu}[j] \quad \text{for } i = 1, \dots, k.$$

Thus, the $\bar{\mu}[i]$ are linearly dependent, as the following matrix relation shows:

$$\begin{bmatrix} \bar{\mu}[i] \\ \vdots \\ \bar{\mu}[k] \end{bmatrix} = \Omega(\rho) \begin{bmatrix} \bar{\mu}[i] \\ \vdots \\ \bar{\mu}[k] \end{bmatrix} \quad \text{where } \Omega(\rho) = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial g_k}{\partial y_1} & \dots & \frac{\partial g_k}{\partial y_k} \end{bmatrix},$$

yielding the linear system: $(\text{Id} - \Omega(\rho))^T [\bar{\mu}[i] \dots \bar{\mu}[k]] = 0$. Here $\Omega(\rho)$ is a stochastic matrix, and from the theory of nonnegative matrices, the rank of $(\text{Id} - \Omega(\rho))$ is $k - 1$. Consequently, the linear system above has a unique solution satisfying $\bar{\mu}[i] + \dots + \bar{\mu}[k] = 1$.

We associate with $\Omega(\rho)$ its dependency digraph D . Namely, we put a directed edge from vertex j to vertex i iff $\frac{\partial g_i}{\partial y_j}(\rho, T_1(\rho), \dots, T_k(\rho)) > 0$. In other words, this directed edge exists iff $\exists C_1 \dots \exists C_k (C_j > 0 \wedge h(C_1, \dots, C_k) = i)$

Property 1. *There is a unique strong component S in D with the property that, for every colour j , there is a directed edge in D from j to some colour in S . S is called the principal component of D .*

This property is the key point to prove the first part of theorem 1:

Theorem 5. *For any system of colouring rules, $\bar{\mu}[i]$ exists for all colours i . Moreover, if S is the principal component of dependency digraph D then*

$$\bar{\mu}[i] > 0 \iff i \in S.$$

Proof. From the property above there exist $A(x)$ and $B(x)$ such that

$$\Omega(x) = \begin{bmatrix} A(x) & C(x) \\ & B(x) \end{bmatrix}$$

and $A(x)$ is an irreducible matrix. Matrix $A(x)$ is indexed by $S = \{1, \dots, s\}$ after renumbering. Since 1 is the largest eigenvalue of $A(\rho)$, from Perron-Frobenius theory (see for instance [7]) there is a unique normalized solution m_1, \dots, m_s of

$$\begin{bmatrix} m_1 \\ \vdots \\ m_s \end{bmatrix} = A(\rho) \begin{bmatrix} m_1 \\ \vdots \\ m_s \end{bmatrix}$$

with $m_1 > 0, \dots, m_s > 0$. Then we just show that 1 cannot be an eigenvalue of $B(\rho)$ and prove that $[m_1, \dots, m_s, 0, \dots, 0]$ is a normalized eigenvector of $\Omega(\rho)$. \square

The colours that do not belong to the principal component S have probabilities that converge exponentially to zero, as the following theorem shows:

Theorem 6. *For any system of colouring rules, if $i \notin S$ then there is some $c > 1$ such that $\mu_n[i] < c^{-n}$. (The same property holds in the unlabeled case).*

Sketch of proof. We prove that for each i such that $i \notin S$, $T_i(x)$ has an analytic continuation on the circle of convergence of $T(x)$, meaning that the radius of convergence of $T_i(x)$, ρ_i , is greater than ρ . Since $\mu_n[i] = t_n^i/t_n$, this leads to desired result. For details, see [8]. \square

5. Existence of $\mu[i]$

We examine here a sufficient condition to ensure the existence of $\mu[i]$ rather than just $\bar{\mu}[i]$. The existence of $\mu[i]$ is conditioned by the presence of a unique singularity on the circle of convergence of $T_i(x)$, indeed:

Lemma 1. *Let $A(x)$ be the irreducible block of matrix $\Omega(x)$. If $\det(A(x) - I) \neq 0$ for all $x \neq \rho$ on the circle $|x| = \rho$ then $\mu[i]$ exists for all $i \in S$. The probabilities $\mu[1], \dots, \mu[s]$ are all strictly positive and form the unique normalized solution of*

$$\begin{bmatrix} \mu[1] \\ \vdots \\ \mu[s] \end{bmatrix} = A(\rho) \begin{bmatrix} \mu[1] \\ \vdots \\ \mu[s] \end{bmatrix}$$

Proof. See [8] \square

If we look again the first example, clearly $\mu[1]$ and $\mu[2]$ do not exist since series $T_{black}(x)$ and $T_{white}(x)$ have two singularities on the circle $|x| = \rho$.

The next theorem is a sufficient criterion on colouring rules to guarantee the convergence of $\mu_n[i]$:

Theorem 7. *Suppose that for each $i \in \{1, \dots, k\}$ there exists at least one pair of rules of the following sort, namely: there exists $C_1 > 1, \dots, C_k > 1$ such that $h(C_1, \dots, C_i - 1, \dots, C_k) = h(C_1, \dots, C_i, \dots, C_k)$. Then $\mu[i]$ exists and $\mu[i] > 0$ for all $i \in S$.*

Sketch of proof. We prove that $\det(I - \Omega(x)) \neq 0$ for all $|x| \leq \rho$ except $x = \rho$, and that each $T_i(x)$ has at most $x = \rho$ as a singularity on the circle $|x| = \rho$. \square

6. Open Problems and Extensions

- Characterize the asymptotic behaviour of $\mu_n[i]$ for general systems of colouring rules;
- Assume φ is a monadic second order sentence. For labeled free trees, McColm [6] proved probability $\mu_n(\varphi)$ satisfied a 0-1 law;
- What happens when we distinguish multiple roots?
- Take unary functions $y = f(x)$. If for some $\epsilon > 0$ and $\gamma > 0$, probability $\mu_n(\varphi) > \epsilon/n^\gamma$ for infinitely many n , is there always a simple asymptotic formula for $\mu_n(\varphi)$?

A partial answer to this last question is that $\mu_n(\varphi)$ converges, and it has been given in the labeled case for $\gamma = 0$ by Compton and Shelah; Woods (also in the unlabeled case) [9]; Łuczak and Thoma [5].

Bibliography

- [1] Compton (Kevin J.). – Application of a Tauberian theorem to finite model theory. *Archiv für Mathematische Logik und Grundlagenforschung*, vol. 25, n° 1-2, 1985, pp. 91–98.
- [2] Compton (Kevin J.). – A logical approach to asymptotic combinatorics. II. Monadic second-order properties. *Journal of Combinatorial Theory. Series A*, vol. 50, n° 1, 1989, pp. 110–131.
- [3] Fagin (Ronald). – Probabilities on finite models. *Journal of Symbolic Logic*, vol. 41, n° 1, 1976, pp. 50–58.
- [4] Glebskiĭ (Y. V.), Kogan (D. I.), Liogon'kiĭ (M. I.), and Talanov (V. A.). – Range of degree and realizability of formulas in the restricted predicate calculus. *Kibernetika (Kiev)*, vol. 5, n° 2, 1969, pp. 17–28. – [Engl. Transl. *Cybernetics*, vol. 5, 142–154 (1972)].

- [5] Łuczak (Tomasz) and Thoma (Luboš). – Convergence of probabilities for the second order monadic properties of a random mapping. *Random Structures & Algorithms*, vol. 11, n° 3, 1997, pp. 277–295.
- [6] McColm (Gregory L.). – MSO asymptotics on random free trees (and random strings). – 1996. Preprint.
- [7] Minc (Henryk). – *Nonnegative matrices*. – John Wiley & Sons Inc., New York, 1988, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, xiv+206p. A Wiley-Interscience Publication.
- [8] Woods (Alan R.). – Coloring rules for finite trees, and probabilities of monadic second order sentences. *Random Structures & Algorithms*, vol. 10, n° 4, 1997, pp. 453–485.
- [9] Woods (Alan R.). – Counting finite models. *The Journal of Symbolic Logic*, vol. 62, n° 3, 1997, pp. 925–949.