Colouring Rules for Finite Trees and Probabilities of Monadic Second Order Sentences

Alan R. Woods
University of Western Australia

March 10, 1998

[summary by Cyril Chabaud]

Abstract

Given a set of colouring rules applying to the vertices of any finite rooted tree, we study the asymptotic behaviour of the probability that an n vertex tree has a given root colour. These results will prove that the fraction of labelled or unlabelled rooted trees satisfying any fixed monadic second-order sentence converge to limiting probabilities.

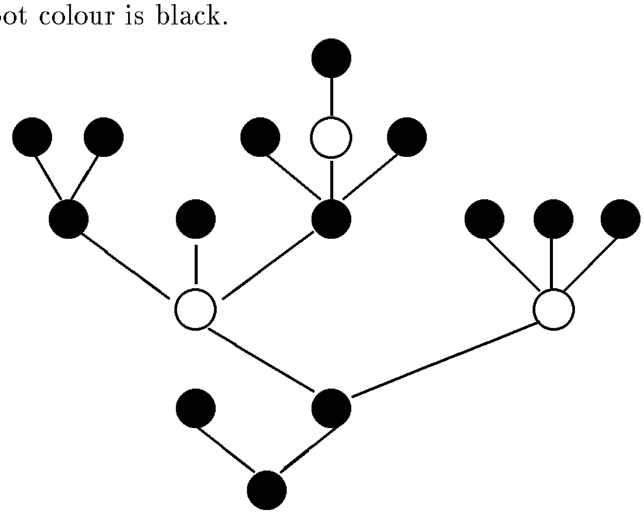
1. Introduction

Given a finite rooted tree and a set of k colours, the vertices are coloured from the leaves to the root according to a set of colouring rules, namely a function $h : \mathbb{N}^k \to \{1, 2, \ldots, k\}$. The colour assigned to a vertex depends only on the number $C_1, \ldots, C_k$ of its immediate predecessors having colour $1, \ldots, k$.

Example. Let

$$h(C_{black}, C_{white}) = \begin{cases} \text{black} & \text{if } C_{black} \text{ is even} \\ \text{white} & \text{if } C_{black} \text{ is odd} \end{cases}$$

be a set of colouring rules. From the definition of h, the leaves of the following tree are coloured black and we find its root colour is black.



Note that the root of a finite rooted tree is black iff the number of its vertices is odd, with the set of colouring rules defined above.

Let $\mu_n[i]$ be the fraction of n vertex labelled trees with root colour i.

Theorem 1. *Let $\mu[i] = \lim_{n\to\infty}\mu_n[i]$ and the corresponding Cesàro limit*

$$\bar{\mu}[i] = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mu_m[i].$$

For any set of colouring rules h, $\bar{\mu}[i]$ exists for all colours $i = 1, \ldots, k$ and either

1. $\bar{\mu}[i] > 0$ *or*
2. $\exists c > 1$ *such that $\mu_n[i] < c^{-n}$ for all sufficiently large n.*

Although the existence of $\mu[i]$ implies the existence of $\bar{\mu}[i]$ in general, the converse needs additional conditions to be true.

2. Applications to Logic

2.1. First Order Logic.

There exists an analogous result for first order sentences about a graph. The language in which these sentences are written contains the usual quantifiers, parentheses and connectives with an additional predicate symbol $E(x, y)$ expressing the fact that vertex x and vertex y are joined by an edge.

Example. The following expression is a first order logic sentence expressing "every vertex has degree 2":

$$\forall x \exists y_1 \exists y_2 (\neg y_1 = y_2 \wedge \forall z (E(x, z) \Leftrightarrow z = y_1 \vee z = y_2)).$$

Fagin [3], Glebskiĭ, Kogan, Liogon'kiĭ and Talanov [4] have proved the following result

Proposition 1. *Let $\mu_n(\varphi)$ be the fraction of n vertex graphs with property φ. For every first order sentence φ about a graph, $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$ exists and $\mu(\varphi) = 0$ or 1.*

That the only possible values are $0, 1$ is a consequence of the fact that graphs have no roots.

2.2. Monadic Second Order Logic.

The situation for monadic second order sentences about a rooted tree is quite different since the language provides a constant symbol R denoting the root, and it can handle sets of vertices using second order variables.

Determining the satisfiability of a monadic second order sentence φ of rank r reduces to finding the root colour of a rooted tree $\mathcal{T}$ for a particular system of colouring rules. Results arising from Compton's method of components [2] establish that if φ is a sentence of rank r, then there exists sentences $\psi_1, \ldots, \psi_k$ of rank r such that:

1. Every finite rooted tree satisfies exactly one ψ_i;
2. Every φ of rank r is equivalent to $\bigvee_{i\in S} \psi_i$ for some set S.

If $\mathcal{T}$ is a rooted tree that has component trees $\mathcal{T}_1, \ldots, \mathcal{T}_m$ that satisfy sentences $\psi_{i_1}, \ldots, \psi_{i_m}$, then there exists a unique i such that $\mathcal{T}$ satisfies ψ_i, and this particular i can be interpreted as the root colour of $\mathcal{T}$. (For details see [8]).

2.3. Boolean Formulas.

Assume we have M boolean variables $x_1, \ldots, x_M$. Then the colours turn out to be the 2^{2^M} boolean functions Ψ_i. The existence of the limiting probability $\mu[i]$ is stated in the following theorem:

Theorem 2. *Let $\mu_n[i]$ be the fraction of formulas of size n which compute the boolean function Ψ_i, $i \in \{1, \ldots, 2^{2^M}\}$. Then $\lim_{n\to\infty} \mu_n[i] = \mu[i]$ exists and $\mu[i] > 0$.*

3. Enumeration of Rooted Trees

3.1. **Labelled Rooted Trees.** We use generating functions methods to determine $\bar{\mu}[i]$ in the labelled case. Note that a similar proof can be done for the unlabelled case.

Let $T(x)$ denote the generating function for labelled rooted trees:

$$T(x) = t_1 x + \frac{t_2}{2!}x^2 + \frac{t_3}{3!}x^3 + \cdots + \frac{t_n}{n!}x^n + \cdots$$

where t_i is the number of i vertex labelled rooted trees. Since this structure is decomposable, we easily obtain a functional equation on $T(x)$ and find:

$$T(x) = xe^{T(x)}.$$

Hence, using Lagrange inversion we get:

$$T(x) = x + \frac{2}{2!}x^2 + \frac{3^2}{3!}x^3 + \cdots + \frac{n^{n-1}}{n!}x^n + \cdots .$$

The radius of convergence of this series is $\rho = 1/e$, $x = \rho$ is the only singularity on the circle of convergence, where there exists a constant h_1 such that $T(x)$ behaves like $1 + h_1\sqrt{\rho - x}$. One can then apply Darboux's theorem and find that t_n behaves asymptotically like $t_n \sim C\rho^{-n}n^{-3/2}$.

3.2. **Labelled Trees with a Particular Root Colour.** Let $T_i(x)$ be the generating function for labelled trees with root colour i,

$$T_i(x) = x \sum_{\substack{M_1,\ldots,M_k \\ h(M_1,\ldots,M_k)=i}} \frac{T_1^{M_1}(x)}{M_1!} \cdots \frac{T_k^{M_k}(x)}{M_k!}.$$

To find $y_i = T_i(x)$ we have to solve the system:

$$\{y_i = g_i(x, y_1, \ldots, y_k)\}_{i \in \{1,\ldots,k\}} \text{ where } g_i(x, y_1, \ldots, y_k) = x \sum_{\substack{M_1,\ldots,M_k \\ h(M_1,\ldots,M_k)=i}} \frac{y_1^{M_1}}{M_1!} \cdots \frac{y_k^{M_k}}{M_k!}.$$

4. Cesàro Probabilities

To determine probability $\bar{\mu}[i]$ we use a partial converse of the following Abelian theorem:

Theorem 3. *Let $b(x) = \sum_{n\geq 0} b_n x^n$, $c(x) = \sum_{n\geq 0} c_n x^n$ and ρ be the radius of convergence of $b(x)$. If $\lim_{n\to\infty} c_n/b_n = \mu$ and $\sum_{n\geq 0} b_n \rho^n$ diverges then:*

$$\lim_{x\to\rho-} c(x)/b(x) = \mu.$$

Setting $c(x) = T_i'(x)$ and $b(x) = T'(x)$, we find that the conditions above are satisfied since $\lim_{x\to\rho-} T'(x) = \infty$. The result is given by the following Tauberian theorem:

Theorem 4 (Compton [1])**.** *Let $b_n \sim Cn^\alpha$, $\alpha > -1$, $b_n \neq 0$ for $n \geq 0$, $c_n = O(b_n)$. If*

$$\lim_{x\to\rho-} c(x)/b(x) = \mu,$$

where ρ is the radius of convergence of $b(x) = \sum_{n\geq 0} b_n x^n$ then:

$$\bar{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{M=1}^{n} \frac{c_M}{b_M} = \mu.$$

Since

$$T_i'(x) = \frac{\partial g_i}{\partial x}(x, T_i(x), \ldots, T_k(x)) + \sum_{j=1}^{k} T_j'(x) \frac{\partial g_i}{\partial y_j}(x, T_i(x), \ldots, T_k(x)),$$

when $x \to \rho-$, we have

$$\bar{\mu}[i] = \sum_{i=1}^{k} \frac{\partial g_i}{\partial y_j} \bar{\mu}[j] \quad \text{for} \quad i = 1, \ldots, k.$$

Thus, the $\bar{\mu}[i]$ are linearly dependent, as the following matrix relation shows:

$$\begin{bmatrix} \bar{\mu}[i] \\ \vdots \\ \bar{\mu}[k] \end{bmatrix} = \Omega(\rho) \begin{bmatrix} \bar{\mu}[i] \\ \vdots \\ \bar{\mu}[k] \end{bmatrix} \text{ where } \Omega(\rho) = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial g_k}{\partial y_1} & \cdots & \frac{\partial g_k}{\partial y_k} \end{bmatrix},$$

yielding the linear system: $(\mathrm{Id} - \Omega(\rho))^T [\bar{\mu}[i] \cdots \bar{\mu}[k]] = 0$. Here $\Omega(\rho)$ is a stochastic matrix, and from the theory of nonnegative matrices, the rank of $(\mathrm{Id} - \Omega(\rho))$ is $k - 1$. Consequently, the linear system above has a unique solution satisfying $\bar{\mu}[i] + \cdots + \bar{\mu}[k] = 1$.

We associate with $\Omega(\rho)$ its dependency digraph D. Namely, we put a directed edge from vertex j to vertex i iff $\frac{\partial g_i}{\partial y_i}(\rho, T_1(\rho), \ldots, T_k(\rho)) > 0$. In other words, this directed edge exists iff $\exists C_1 \cdots \exists C_k (C_j > 0 \wedge h(C_1, \ldots, C_k) = i)$

Property 1. *There is a unique strong component S in D with the property that, for every colour j, there is a directed edge in D from j to some colour in S. S is called the principal component of D.*

This property is the key point to prove the first part of theorem 1:

Theorem 5. *For any system of colouring rules, $\bar{\mu}[i]$ exists for all colours i. Moreover, if S is the principal component of dependency digraph D then*

$$\bar{\mu}[i] > 0 \Longleftrightarrow i \in S.$$

Proof. From the property above there exist $A(x)$ and $B(x)$ such that

$$\Omega(x) = \begin{bmatrix} A(x) & C(x) \\ & B(x) \end{bmatrix}$$

and $A(x)$ is an irreducible matrix. Matrix $A(x)$ is indexed by $S = \{1, \ldots, s\}$ after renumbering. Since 1 is the largest eigenvalue of $A(\rho)$, from Perron-Frobenius theory (see for instance [7]) there is a unique normalized solution $m_1, \ldots, m_s$ of

$$\begin{bmatrix} m_1 \\ \vdots \\ m_s \end{bmatrix} = A(\rho) \begin{bmatrix} m_1 \\ \vdots \\ m_s \end{bmatrix}$$

with $m_1 > 0, \ldots, m_s > 0$. Then we just show that 1 cannot be an eigenvalue of $B(\rho)$ and prove that $[m_1, \ldots, m_s, 0, \ldots, 0]$ is a normalized eigenvector of $\Omega(\rho)$. □

The colours that do not belong to the principal component S have probabilities that converge exponentially to zero, as the following theorem shows:

Theorem 6. *For any system of colouring rules, if $i \notin S$ then there is some $c > 1$ such that $\mu_n[i] < c^{-n}$. (The same property holds in the unlabeled case).*

Sketch of proof. We prove that for each i such that $i \notin S$, $T_i(x)$ has an analytic continuation on the circle of convergence of $T(x)$, meaning that the radius of convergence of $T_i(x)$, ρ_i, is greater than ρ. Since $\mu_n[i] = t_n^i / t_n$, this leads to desired result. For details, see [8]. □

5. Existence of $\mu[i]$

We examine here a sufficient condition to ensure the existence of $\mu[i]$ rather than just $\bar{\mu}[i]$. The existence of $\mu[i]$ is conditioned by the presence of a unique singularity on the circle of convergence of $T_i(x)$, indeed:

Lemma 1. *Let $A(x)$ be the irreducible block of matrix $\Omega(x)$. If $\det(A(x) - I) \neq 0$ for all $x \neq \rho$ on the circle $|x| = \rho$ then $\mu[i]$ exists for all $i \in S$. The probabilities $\mu[1], \ldots, \mu[s]$ are all strictly positive and form the unique normalized solution of*

$$\begin{bmatrix} \mu[1] \\ \vdots \\ \mu[s] \end{bmatrix} = A(\rho) \begin{bmatrix} \mu[1] \\ \vdots \\ \mu[s] \end{bmatrix}$$

Proof. See [8] □

If we look again the first example, clearly $\mu[1]$ and $\mu[2]$ do not exist since series $T_{black}(x)$ and $T_{white}(x)$ have two singularities on the circle $|x| = \rho$.

The next theorem is a sufficient criterion on colouring rules to guarantee the convergence of $\mu_n[i]$:

Theorem 7. *Suppose that for each $i \in \{1, \ldots, k\}$ there exists at least one pair of rules of the following sort, namely: there exists $C_1 > 1, \ldots, C_k > 1$ such that $h(C_1, \ldots, C_i - 1, \ldots, C_k) = h(C_1, \ldots, C_i, \ldots, C_k)$. Then $\mu[i]$ exists and $\mu[i] > 0$ for all $i \in S$.*

Sketch of proof. We prove that $\det(I - \Omega(x)) \neq 0$ for all $|x| \leq \rho$ except $x = \rho$, and that each $T_i(x)$ has at most $x = \rho$ as a singularity on the circle $|x| = \rho$. □

6. Open Problems and Extensions

- Characterize the asymptotic behaviour of $\mu_n[i]$ for general systems of colouring rules;
- Assume φ is a monadic second order sentence. For labeled free trees, McColm [6] proved probability $\mu_n(\varphi)$ satisfied a 0-1 law;
- What happens when we distinguish multiple roots?
- Take unary functions $y = f(x)$. If for some $\epsilon > 0$ and $\gamma > 0$, probability $\mu_n(\varphi) > \epsilon/n^\gamma$ for infinitely many n, is there always a simple asymptotic formula for $\mu_n(\varphi)$?

A partial answer to this last question is that $\mu_n(\varphi)$ converges, and it has been given in the labeled case for $\gamma = 0$ by Compton and Shelah; Woods (also in the unlabeled case) [9]; Luczak and Thoma [5].

Bibliography

[1] Compton (Kevin J.). – Application of a Tauberian theorem to finite model theory. *Archiv für Mathematische Logik und Grundlagenforschung*, vol. 25, n° 1-2, 1985, pp. 91–98.

[2] Compton (Kevin J.). – A logical approach to asymptotic combinatorics. II. Monadic second-order properties. *Journal of Combinatorial Theory. Series A*, vol. 50, n° 1, 1989, pp. 110–131.

[3] Fagin (Ronald). – Probabilities on finite models. *Journal of Symbolic Logic*, vol. 41, n° 1, 1976, pp. 50–58.

[4] Glebskiĭ (Y. V.), Kogan (D. I.), Liogon'kiĭ (M. I.), and Talanov (V. A.). – Range of degree and realizability of formulas in the restricted predicate calculus. *Kibernetika (Kiev)*, vol. 5, n° 2, 1969, pp. 17–28. – [Engl. Transl. *Cybernetics, vol. 5,* 142–154 (1972)].

[5] Łuczak (Tomasz) and Thoma (Luboš). – Convergence of probabilities for the second order monadic properties of a random mapping. *Random Structures & Algorithms*, vol. 11, n° 3, 1997, pp. 277–295.
[6] McColm (Gregory L.). – MSO asymptotics on random free trees (and random strings). – 1996. Preprint.
[7] Minc (Henryk). – *Nonnegative matrices.* – John Wiley & Sons Inc., New York, 1988, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, xiv+206p. A Wiley-Interscience Publication.
[8] Woods (Alan R.). – Coloring rules for finite trees, and probabilities of monadic second order sentences. *Random Structures & Algorithms*, vol. 10, n° 4, 1997, pp. 453–485.
[9] Woods (Alan R.). – Counting finite models. *The Journal of Symbolic Logic*, vol. 62, n° 3, 1997, pp. 925–949.