Colouring Rules for Finite Trees
and Probabilities of Monadic Second Order Sentences

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Abstract

Given a set of colouring rules applying to the vertices of any finite rooted tree, we study
the asymptotic behaviour of the probability that an n vertex tree has a given root colour.
These results will prove that the fraction of labelled or unlabelled rooted trees satisfying
any fixed monadic second-order sentence converge to limiting probabilities.

1. Introduction

Given a finite rooted tree and a set of k colours, the vertices are coloured from the leaves to the
root according to a set of colouring rules, namely a function $h : N^k \to \{1, 2, \ldots, k\}$. The colour
assigned to a vertex depends only on the number $C_1, \ldots, C_k$ of its immediate predecessors having
colour 1, \ldots, k.

Example. Let

$$h(C_{\text{black}}, C_{\text{white}}) = \begin{cases} 
\text{black} & \text{if } C_{\text{black}} \text{ is even} \\
\text{white} & \text{if } C_{\text{black}} \text{ is odd}
\end{cases}$$

be a set of colouring rules. From the definition of $h$, the leaves of the following tree are coloured
black and we find its root colour is black.

![Tree diagram]
Note that the root of a finite rooted tree is black if the number of its vertices is odd, with the set of colouring rules defined above.

Let \( \mu_n[i] \) be the fraction of \( n \) vertex labelled trees with root colour \( i \).

**Theorem 1.** Let \( \mu[i] = \lim_{n \to \infty} \mu_n[i] \) and the corresponding Cesàro limit

\[
\bar{\mu}[i] = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mu_m[i].
\]

For any set of colouring rules \( h \), \( \bar{\mu}[i] \) exists for all colours \( i = 1, \ldots, k \) and either

1. \( \bar{\mu}[i] > 0 \) or
2. \( \exists c > 1 \) such that \( \mu_n[i] < c^{-n} \) for all sufficiently large \( n \).

Although the existence of \( \mu[i] \) implies the existence of \( \bar{\mu}[i] \) in general, the converse needs additional conditions to be true.

2. Applications to Logic

2.1. First Order Logic. There exists an analogous result for first order sentences about a graph. The language in which these sentences are written contains the usual quantifiers, parentheses and connectives with an additional predicate symbol \( E(x, y) \) expressing the fact that vertex \( x \) and vertex \( y \) are joined by an edge.

**Example.** The following expression is a first order logic sentence expressing “every vertex has degree 2”:

\[
\forall x \exists y_1 \exists y_2 (-y_1 = y_2 \land \forall z (E(x, z) \iff z = y_1 \lor z = y_2)).
\]

Fagin [3], Glebskiĭ, Kogan, Liogon’kiĭ and Talanov [4] have proved the following result

**Proposition 1.** Let \( \mu_n(\varphi) \) be the fraction of \( n \) vertex graphs with property \( \varphi \). For every first order sentence \( \varphi \) about a graph, \( \mu(\varphi) = \lim_{n \to \infty} \mu_n(\varphi) \) exists and \( \mu(\varphi) = 0 \) or 1.

That the only possible values are 0, 1 is a consequence of the fact that graphs have no roots.

2.2. Monadic Second Order Logic. The situation for monadic second order sentences about a rooted tree is quite different since the language provides a constant symbol \( R \) denoting the root, and it can handle sets of vertices using second order variables.

Determining the satisfiability of a monadic second order sentence \( \varphi \) of rank \( r \) reduces to finding the root colour of a rooted tree \( T \) for a particular system of colouring rules. Results arising from Compton’s method of components [2] establish that if \( \varphi \) is a sentence of rank \( r \), then there exists sentences \( \psi_1, \ldots, \psi_k \) of rank \( r \) such that:

1. Every finite rooted tree satisfies exactly one \( \psi_i \);
2. Every \( \varphi \) of rank \( r \) is equivalent to \( \bigvee_{i \in S} \psi_i \) for some set \( S \).

If \( T \) is a rooted tree that has component trees \( T_1, \ldots, T_m \) that satisfy sentences \( \psi_i_1, \ldots, \psi_i_m \), then there exists a unique \( i \) such that \( T \) satisfies \( \psi_i \), and this particular \( i \) can be interpreted as the root colour of \( T \). (For details see [8]).

2.3. Boolean Formulas. Assume we have \( M \) boolean variables \( x_1, \ldots, x_M \). Then the colours turn out to be the \( 2^{2^M} \) boolean functions \( \Psi_i \). The existence of the limiting probability \( \mu[i] \) is stated in the following theorem:

**Theorem 2.** Let \( \mu_n[i] \) be the fraction of formulas of size \( n \) which compute the boolean function \( \Psi_i \), \( i \in \{1, \ldots, 2^2^M \} \). Then \( \lim_{n \to \infty} \mu_n[i] = \mu[i] \) exists and \( \mu[i] > 0 \).
3. Enumeration of Rooted Trees

3.1. Labelled Rooted Trees. We use generating functions methods to determine $\beta[i]$ in the labelled case. Note that a similar proof can be done for the unlabelled case.

Let $T(x)$ denote the generating function for labelled rooted trees:

$$T(x) = t_1 x + \frac{t_2 x^2}{2!} + \frac{t_3 x^3}{3!} + \cdots + \frac{t_n x^n}{n!} + \cdots$$

where $t_i$ is the number of $i$ vertex labelled rooted trees. Since this structure is decomposable, we easily obtain a functional equation on $T(x)$ and find:

$$T(x) = xe^{T(x)}.$$

Hence, using Lagrange inversion we get:

$$T(x) = x + \frac{2}{2!} x^2 + \frac{3^2}{3!} x^3 + \cdots + \frac{n^{n-1}}{n!} x^n + \cdots.$$

The radius of convergence of this series is $\rho = 1/e$, $x = \rho$ is the only singularity on the circle of convergence, where there exists a constant $h_1$ such that $T(x)$ behaves like $1 + h_1 \sqrt{x - x}$. One can then apply Darboux’s theorem and find that $t_n$ behaves asymptotically like $t_n \sim C\rho^{-n/2}$.

3.2. Labelled Trees with a Particular Root Colour. Let $T_i(x)$ be the generating function for labelled trees with root colour $i$,

$$T_i(x) = x \sum_{M_1, \ldots, M_k \in \{1, \ldots, n\}} \frac{T_1^{M_1}(x)}{M_1!} \cdots \frac{T_k^{M_k}(x)}{M_k!}.$$

To find $y_i = T_i(x)$ we have to solve the system:

$$\{y_i = g_i(x, y_1, \ldots, y_k)\}_{i \in \{1, \ldots, n\}}$$

where $g_i(x, y_1, \ldots, y_k) = x \sum_{M_1, \ldots, M_k \in \{1, \ldots, n\}} \frac{y_1^{M_1}}{M_1!} \cdots \frac{y_k^{M_k}}{M_k!}$.

4. Cesàro Probabilities

To determine probability $\beta[i]$ we use a partial converse of the following Abelian theorem:

**Theorem 3.** Let $b(x) = \sum_{n \geq 0} b_n x^n$, $c(x) = \sum_{n \geq 0} c_n x^n$ and $\rho$ be the radius of convergence of $b(x)$. If $\lim_{n \to \infty} c_n / b_n = \mu$ and $\sum_{n \geq 0} b_n \rho^n$ diverges then:

$$\lim_{x \to \rho^-} c(x)/b(x) = \mu.$$

Setting $c(x) = T'(x)$ and $b(x) = T(x)$, we find that the conditions above are satisfied since $\lim_{x \to \rho^-} T'(x) = \infty$. The result is given by the following Tauberian theorem:

**Theorem 4 (Compton [1]).** Let $b_n \sim Cn^{\alpha}$, $\alpha > -1$, $b_n \neq 0$ for $n \geq 0$, $c_n = O(b_n)$. If

$$\lim_{x \to \rho^-} c(x)/b(x) = \mu,$$

where $\rho$ is the radius of convergence of $b(x) = \sum_{n \geq 0} b_n x^n$ then:

$$\beta = \lim_{n \to \infty} \frac{1}{n} \sum_{M=1}^{n} \frac{c_M}{b_M} = \mu.$$
Since
\[ T_i'(x) = \frac{\partial g_i}{\partial x}(x, T_1(x), \ldots, T_k(x)) + \sum_{j=1}^{k} T_j'(x) \frac{\partial g_i}{\partial y_j}(x, T_1(x), \ldots, T_k(x)), \]
when \( x \to \rho - \), we have
\[ \bar{\mu}[i] = \sum_{j=1}^{k} \frac{\partial g_i}{\partial y_j} \bar{\mu}[j] \quad \text{for} \quad i = 1, \ldots, k. \]
Thus, the \( \bar{\mu}[i] \) are linearly dependent, as the following matrix relation shows:
\[
\begin{bmatrix}
\bar{\mu}[i] \\
\vdots \\
\bar{\mu}[k]
\end{bmatrix}
= \Omega(\rho) \begin{bmatrix}
\bar{\mu}[1] \\
\vdots \\
\bar{\mu}[k]
\end{bmatrix}
\quad \text{where} \quad \Omega(\rho) = \begin{bmatrix}
\frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_k} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_k}{\partial y_1} & \cdots & \frac{\partial g_k}{\partial y_k}
\end{bmatrix},
\]
yielding the linear system: \((\text{Id} - \Omega(\rho)) \bar{T}[\bar{\mu}[1] \cdots \bar{\mu}[k]] = 0\). Here \( \Omega(\rho) \) is a stochastic matrix, and from the theory of nonnegative matrices, the rank of \((\text{Id} - \Omega(\rho))\) is \( k - 1 \). Consequently, the linear system above has a unique solution satisfying \( \bar{\mu}[i] + \cdots + \bar{\mu}[k] = 1 \).

We associate with \( \Omega(\rho) \) its dependency digraph \( D \). Namely, we put a directed edge from vertex \( j \) to vertex \( i \) iff \( \frac{\partial g_i}{\partial y_j}(\rho, T_1(\rho), \ldots, T_k(\rho)) > 0 \). In other words, this directed edge exists iff \( \exists C_1 \cdots \exists C_k(C_j > 0 \land h(C_1, \ldots, C_k) = i) \).

**Property 1.** There is a unique strong component \( S \) in \( D \) with the property that, for every colour \( j \), there is a directed edge in \( D \) from \( j \) to some colour in \( S \). \( S \) is called the principal component of \( D \).

This property is the key point to prove the first part of theorem 1:

**Theorem 5.** For any system of colouring rules, \( \bar{\mu}[i] \) exists for all colours \( i \). Moreover, if \( S \) is the principal component of dependency digraph \( D \) then
\[ \bar{\mu}[i] > 0 \iff i \in S. \]

**Proof.** From the property above there exist \( A(x) \) and \( B(x) \) such that
\[
\Omega(x) = \begin{bmatrix}
A(x) & C(x) \\
B(x)
\end{bmatrix}
\]
and \( A(x) \) is an irreducible matrix. Matrix \( A(x) \) is indexed by \( S = \{1, \ldots, s\} \) after renumbering. Since 1 is the largest eigenvalue of \( A(x) \), from Perron-Frobenius theory (see for instance [7]) there is a unique normalized solution \( m_1, \ldots, m_s \) of
\[
\begin{bmatrix}
m_1 \\
\vdots \\
m_s
\end{bmatrix}
= A(\rho) \begin{bmatrix}
m_1 \\
\vdots \\
m_s
\end{bmatrix}
\]
with \( m_1 > 0, \ldots, m_s > 0 \). Then we just show that 1 cannot be an eigenvalue of \( B(\rho) \) and prove that \( [m_1, \ldots, m_s, 0, \ldots, 0] \) is a normalized eigenvector of \( \Omega(\rho) \).

The colours that do not belong to the principal component \( S \) have probabilities that converge exponentially to zero, as the following theorem shows:

**Theorem 6.** For any system of colouring rules, if \( i \notin S \) then there is some \( c > 1 \) such that
\[ \mu_n[i] < c^{-n}. \] (The same property holds in the unlabeled case).
Sketch of proof. We prove that for each $i$ such that $i \notin S$, $T_i(x)$ has an analytic continuation on the circle of convergence of $T(x)$, meaning that the radius of convergence of $T_i(x)$, $\rho_i$, is greater than $\rho$. Since $\mu_n[i] = t_n/i_n$, this leads to desired result. For details, see [8].

5. Existence of $\mu[i]$

We examine here a sufficient condition to ensure the existence of $\mu[i]$ rather than just $\beta[i]$. The existence of $\mu[i]$ is conditioned by the presence of a unique singularity on the circle of convergence of $T_i(x)$, indeed:

**Lemma 1.** Let $A(x)$ be the irreducible block of matrix $\Omega(x)$. If $\det(A(x) - I) \neq 0$ for all $x \neq \rho$ on the circle $|x| = \rho$ then $\mu[i]$ exists for all $i \in S$. The probabilities $\mu[1], \ldots, \mu[s]$ are all strictly positive and form the unique normalized solution of

$$
\begin{bmatrix}
\mu[1] \\
\vdots \\
\mu[s]
\end{bmatrix} = A(\rho)
\begin{bmatrix}
\mu[1] \\
\vdots \\
\mu[s]
\end{bmatrix}
$$

**Proof.** See [8] □

If we look again the first example, clearly $\mu[1]$ and $\mu[2]$ do not exist since series $T_{black}(x)$ and $T_{white}(x)$ have two singularities on the circle $|x| = \rho$.

The next theorem is a sufficient criterion on coloring rules to guarantee the convergence of $\mu_n[i]$:  

**Theorem 7.** Suppose that for each $i \in \{1, \ldots, k\}$ there exists at least one pair of rules of the following sort, namely: there exists $C_1 > 1, \ldots, C_k > 1$ such that $h(C_1, \ldots, C_i - 1, \ldots, C_k) = h(C_1, \ldots, C_i, \ldots, C_k)$. Then $\mu[i]$ exists and $\mu[i] > 0$ for all $i \in S$.

**Sketch of proof.** We prove that $\det(I - \Omega(x)) \neq 0$ for all $|x| \leq \rho$ except $x = \rho$, and that each $T_i(x)$ has at most $x = \rho$ as a singularity on the circle $|x| = \rho$. □

6. Open Problems and Extensions

- Characterize the asymptotic behaviour of $\mu_n[i]$ for general systems of coloring rules;
- Assume $\varphi$ is a monadic second order sentence. For labeled free trees, McColm [6] proved probability $\mu_n(\varphi)$ satisfied a 0-1 law;
- What happens when we distinguish multiple roots?
- Take unary functions $y = f(x)$. If for some $\epsilon > 0$ and $\gamma > 0$, probability $\mu_n(\varphi) > \epsilon/n^\gamma$ for infinitely many $n$, is there always a simple asymptotic formula for $\mu_n(\varphi)$?

A partial answer to this last question is that $\mu_n(\varphi)$ converges, and it has been given in the labeled case for $\gamma = 0$ by Compton and Shelah; Woods (also in the unlabeled case) [9]; Luczak and Thoma [5].

Bibliography