Two Not–That–Dull Functional Equations Arising in the Analysis of Algorithms*

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Outline of the Talk

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   (a) Formulation of the Problem
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Recurrences and Functional Equations

In this talk, we discuss two types of functional equations arising in the analysis of GENERALIZED DIGITAL SEARCH TREES and the ASYMMETRIC LEADER ELECTION algorithm.

Generalized Digital Search Trees

Let $b > 1$. For a given sequence a_n and a constant u

$$x_{n+b} = a_n + u \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (x_k + x_{n-k}) \qquad n \geq 0$$

with some initial condition. The Poisson transform $\widetilde{X}(z) = \sum_{n\geq 0} x_n \frac{z^n}{n!} e^{-z}$ of x_n satisfies the following differential-functional equation

$$\sum_{i=0}^{b} \binom{b}{i} \frac{\partial^i \widetilde{X}(z)}{\partial z^i} = \widetilde{A}(z) + u(\widetilde{X}(pz) + \widetilde{X}(qz))$$

where $q = 1 - p$.

(Flajolet&Richmond, RSA'92 for $b > 1$ and $p = \frac{1}{2}$).

Recurrences and Functional Equations

Asymmetric Leader Election Algorithm
Give x_0 and x_1, let

$$x_n = 1 + q^n x_n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} x_k \ , \qquad n \geq 2 \ .$$

Its Poisson transform satisfies

$$\widetilde{X}(z) = \widetilde{X}(pz) + \widetilde{X}(qz)e^{-pz} + 1 - (1+z)e^{-z},$$

or in general

$$f(z) = f(pz) + f(qz)e^{-pz} + a(z)$$

for some function $a(z)$.

In general, when studying distribution, we must deal with the following functional equation:

$$f_{k+1}(z) = f_k(pz) + e^{-pz} f_k(qz) \quad k \geq 1 \ .$$

(Fayolle, Flajolet, Hofri, AAP 1984.)

Motivation: Generalized Lempel-Ziv Parsing Scheme

The original Lempel-Ziv parsing scheme partitions a sequence of symbols into variable phrases such that the next phrase is the shortest phrase not seen in the past.

For example,

$$\mathtt{ababbababaaaaaaaaac}$$

is parsed into

$$\mathtt{(a)(b)(ab)(ba)(bab)(aa)(aaa)(aaaa)(c)}$$

and its code becomes:

$$\mathtt{0a0b1b2a4b1a6a7a0c}$$

which requires 54 bits.

(The Lempel–Ziv code consists of pairs $(\mathtt{pointer}, \mathtt{symbol})$ each pair being a pointer to the previous occurrence of the prefix of the phrase and the last symbol of the phrase).

Generalized Lempel-Ziv Parsing Scheme

Let $b \in \{1, 2, \ldots\}$ be a parameter. A sequence is partitioned in phrases such that the next phrase is the SHORTEST phrase seen in the past by *at most* $b - 1$ phrases. ($b = 1$ corresponds to the original Lempel-Ziv algorithm).

Let $b = 2$, and consider the string $ababbbabbaaaba$ which is parsed and coded as follows:

Phrase:	1	2	-	-	3	4	-	5
Sequence:	(a)	(b)	(a)	(b)	(bb)	(ab)	(bb)	(aa)
Code:	0a	0b	1	2	2b	1b	3	1a

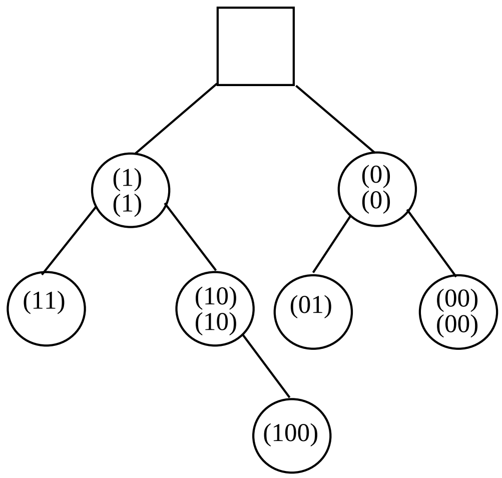
For example, for $b = 2$ `ababbababaaaaaaaaac` is parsed into

`(a)(b)(a)(b)(ba)(ba)(baa)(aa)(aa)(aaa)(c)`

`0a0b122a33a1a55a0c`

which requires 47 bits. If we do the same with $b = 3$ we need only 46 bits, while for $b = 4$ we need again 52 bits.

Generalized Digital Search Tree



A b-digital search tree representation with $b = 2$ of the generalized Lempel-Ziv parsing

$$(1)(1)(0)(0)(10)(10)(00)(100)(01)(00)(01)(11)$$

of the string 11001010001000010011.

b-Digital Search Tree Model

DIGITAL TREE MODEL:
A b-digital search tree is built from FIXED number, say m, of INDEPENDENT strings such that every node can store up to b strings. Strings are generated according to the Bernoulli model:

symbols are generated in an independent manner with "0" and "1" occurring respectively with probability p and $q = 1 - p$.
If $p = q = 0.5$, the the Bernoulli model is called *symmetric*, otherwise it is *asymmetric*.

Parameters:
$S_m(b)$ – size of the tree (i.e., number of nodes),
D_m – typical depth, that is, length of a path from the root to a randomly selected string,
D_m^i – depth of the ith string (path length from the root to the node containing the ith string,
L_m – (total or internal) path length

$$L_m = \sum_{i=1}^{m} D_m^i \ .$$

Depth in the Digital Tree Model

Recall the definition of the typical depth:

$$P\{D_m = k\} = \sum_{i=1}^{m} \frac{P\{D_m^i = k\}}{m} \ .$$

Average Profile: Let B_m^k be the average number of items (strings) on level k of a randomly built b-digital search tree. Then:

$$\Pr\{D_m = k\} = \frac{B_m^k}{m} \ .$$

Define the generating function $B_m(u)$ of B_m^k as

$$B_m(u) = \sum_{k \geq 0} B_m^k u^k$$

for u complex.

Recurrence Equation

Due to a recursive structure of digital trees, we have:

$$B_{m+b}(u) = b + u \sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i} \left(B_i(u) + B_{m-i}(u) \right)$$

with initial conditions

$$B_i(u) = i \qquad \text{for} \qquad i = 0, 1, \ldots, b$$

Observe that $B(1) = m$.
(For $b > 1$ and $p = q = 0.5$, Flajolet and Richmond solved the above using Harmonic Sum Formula.)

Poissonization

A standard "trick" in the probabilistic (and analytical) toolkit says:
If you cannot solve the model at hand, **poissonize** *it, that is, replace the deterministic input by a Poisson input.*

In our case, we define a new generating function called the Poisson generating function:

$$\widetilde{B}(u,z) = \sum_{i=0}^{\infty} B_i(u) \frac{z^i}{i!} e^{-z} \ .$$

Then, our basic recurrence becomes:

$$(1 + \frac{\partial}{\partial z})^b \widetilde{B}(u,z) = b + u \left(\widetilde{B}(u,pz) + \widetilde{B}(u,qz) \right)$$

where

$$(1 + \frac{\partial}{\partial z})^b f(z) =^{\text{def}} \sum_{i=0}^{b} \binom{b}{i} \frac{\partial^i f(z)}{\partial z^i} \ .$$

Main Results: Digital Tree Model

Theorem 1. *Under the* **asymmetric** *Bernoulli model:*

$$\begin{aligned} ED_m &= \frac{1}{h_1}\log m + \frac{1}{h_1}\left(\frac{h_2}{2h_1} + \gamma - 1 - H_{b-1} - \Delta(b,p)\right. \\ &+ \left.\delta(\log_2 m)\right) + O\left(\frac{\log m}{m}\right) \\ \mathrm{Var}\ D_m &= \frac{h_2 - h_1^2}{h_1^3}\log m + O(1) \end{aligned}$$

where $h_1 = -p\log p - q\log q$ *is the entropy,* $h_2 = p\log^2 p + q\log^2 q$*, and* $\gamma = 0.577\ldots$ *is Euler constant, while* $H_{b-1} = \sum_{i=1}^{b-1}\frac{1}{i}$*,* $H_0 = 0$ *is the harmonic sum. The constant* $\Delta(b,p)$ *can be computed as follows*

$$\Delta(b,p) = \sum_{n=2b+1}^{\infty} \bar{f}_n \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\ldots(n-i-1)}$$

where $\bar{f}_n$ *is given recursively by*

$$\begin{cases} f_{m+b} = m + \sum_{i=0}^{m}\binom{m}{i}p^i q^{m-i}(f_i + f_{m-i}), \ for\ m > 0, \\ f_0 = f_1 = \ldots = f_b = 0, \\ \bar{f}_{m+b} = f_{m+b} - m > 0, \ m \geq b+1. \end{cases}$$

Finally, $\delta(\log_2 m)$ *is a fluctuating function with a small amplitude when* $\log p/\log q$ *is rational, and* $\delta(\log_2 m) \equiv 0$ *otherwise.*

(ii) *Let* $G_m(u)$ *be the ordinary generating function of* D_m *(i.e.,* $G_m(u) = Eu^{D_m}$*),* $\mu_m = ED_m$*, and* $\sigma_m = \sqrt{\mathrm{Var}\ D_m}$*. Then, for a complex* τ

$$e^{-\tau\mu_m/\sigma_m} G_m(e^{\tau/\sigma_m}) = e^{\frac{\tau^2}{2}}\left(1 + O\left(\frac{1}{\sqrt{\log m}}\right)\right)$$

Thus, the limiting distribution of $\frac{D_m - \mu_m}{\sigma_m}$ *is normal, and it converges in moments to the appropriate moments of the standard normal distribution.*

(iii) *There exist positive constants* A *and* $\alpha < 1$ *such that*

$$\Pr\left\{\left|\frac{D_m - c_1 \log m}{\sqrt{c_2 \log m}}\right| > k\right\} \leq A\alpha^k$$

uniformly in k *for large* m*.*

Numerical Values

Table 1: Numerical values of $\Delta(b,p)$ and $ED_m - \frac{1}{h_1}\log m$ for $p = 0.3$

b	$\Delta(b,p)$	$ED_m - \frac{1}{h_1}(\log m - \delta(m,b))$
1	1.25	— 2.04
2	0.96	— 3.20
3	0.91	— 3.94
5	0.83	— 4.76
8	0.76	— 5.48
20	0.60	— 6.78
50	0.36	— 7.91
90	0.12	— 8.49

Main Results– Symmetric Digital Tree Model

Remark: In the symmetric Bernoulli model ($p = q = 0.5$) the limiting distribution does not exist! For example, Louchard and Szpankowski (IT, 1995) proved that for $b = 1$ we have:

Let $Q_k = \prod_{j=1}^{k}(1 - 2^{-j})$, *and define*

$$\psi(m) = \log_2 m - \lfloor \log_2 m \rfloor \ .$$

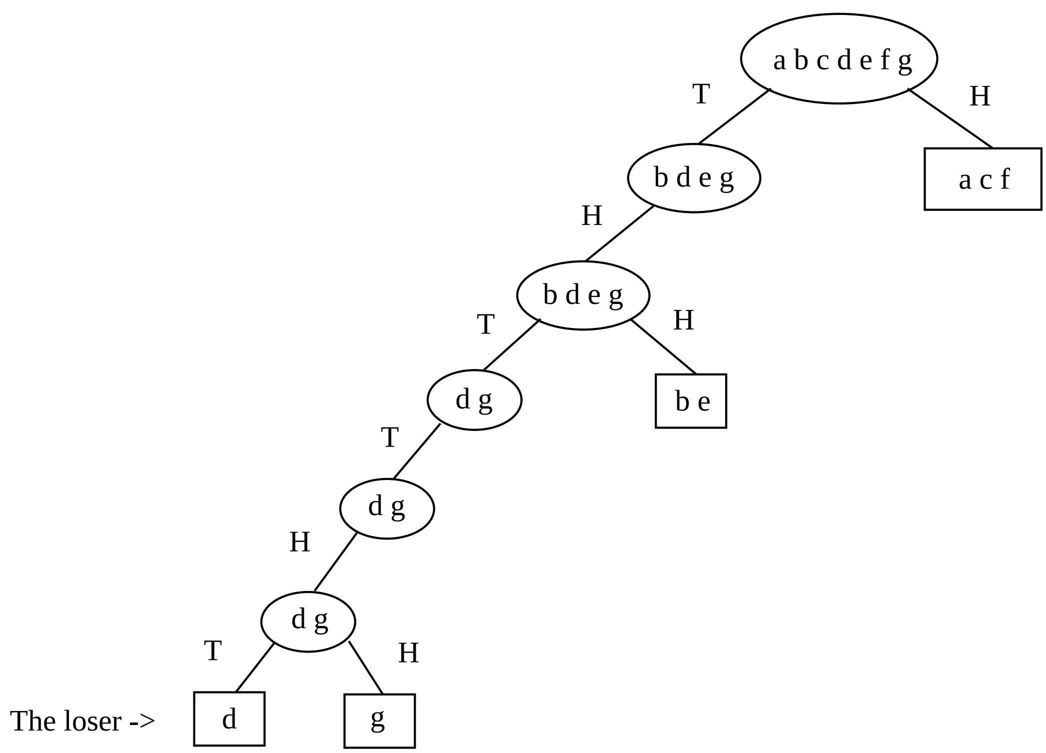
Then, for the symmetric Bernoulli we obtain for any integer K

$$\lim_{m\to\infty} \left| \Pr\{D_m \le \log_2 m + K\} - 2^{K-\psi(m)} \left(1 + \frac{1}{2Q_\infty} \sum_{i=0}^{\infty} (-1)^{i+1} \frac{2^{-i(i+1)/2}}{Q_i} e^{-2^{-(K-\psi(m)-1-i)}} \right) \right| = 0 \ .$$

The function $\psi(m)$ *is dense in* $[0, 1]$ *but not uniformly dense, thus the limiting distribution of* D_m *does not exist.*

Asymmetric Leader Election Algorithm

A set of distributed objects (people, computers, etc.) try to identify one object as their leader. The election process is randomized, that is, at every stage of the algorithm those objects that survived so far flip a *biased* coin, and those who received, say a tail, survive for the next round. The process continues until only one objects remains. Let p be the probability of survival.



Recurrences and Functional Equations

Let H_n be the number of rounds needed to identify the leader or equivalently the height in the incomplete trie. Let $G_n(u) = \mathbf{E}u^{H_n} = \sum_{k\geq 0} \mathbf{P}(H_n = k)z^k$ be the probability generating function of H_n Then, $G_1(u) = 1$ and for $n \geq 2$

$$G_n(u) = u \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} G_k(u) + uq^n G_n(u) \ .$$

Let now $x_n = \mathbf{E}H_n$ and $w_n = \mathbf{E}H_n(H_n - 1)$. Observing that $x_n = G'_n(1)$ and $w_n = G''_n(1)$, we derive

$$\begin{aligned} x_n &= 1 + q^n x_n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} x_k \ , \qquad n \geq 2, \\ w_n &= 2(x_n - 1) + q^n w_n + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} w_k \ , \qquad n \geq 2 \ , \end{aligned}$$

with $x_0 = x_1 = w_0 = w_1 = 0$.

Prodinger, *Disc. Math*, 1993 (only mean value),
Fill, Mahmoud, Szpankowski, *Appl. Ann. Probab*, 1997 (symmetric case).

Main Results

Theorem 2. *Let* $P := 1/p$ *and* $\chi_k := 2\pi i k/\ln P$. *Then:*

(i) *The mean* $\mathbf{E}H_n$ *of the height admits the following asymptotic formula*

$$\mathbf{E}H_n = \log_P n + \frac{1}{2} - \frac{1-\gamma-T_1^*(0)}{\ln P} + \delta_1(\log_P n) + O(1/n)$$

where $\gamma = 0.577\ldots$ *is the Euler constant, and*

$$T_1^*(0) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n} ,$$

where x_n *must be computed from the original recurrence. The function* $\delta_1(x)$ *is periodic function of small magnitude (e.g., for* $p = 0.5$ *one proves* $|\delta_1(x)| \leq 2 \times 10^{-5}$*) given by* $\delta_1(x) = -\sum_{k\neq 0} \alpha_k e^{-2\pi i k x}$ *where*

$$\alpha_k = \frac{(1+\chi_k)\Gamma(\chi_k) - T_1^*(\chi_k)}{\ln P} ,$$

$\Gamma(s)$ *is the Euler gamma function and* $T_1^*(s)$ *is discussed later.*

(ii) *The variance satisfies*

$$
\begin{aligned}
\mathrm{Var}\, H_n &= \frac{\pi^2/6 - 1 + 2(1-\gamma)T_1^*(0) - 2T_1^{*\prime}(0) - (T_1^*(0))^2}{\ln^2 P} \\
&+ \frac{2T_1^*(0) + T_2^*(0)}{\ln P} + \frac{1}{12} - [\delta_1^2]_0 + \delta_2(\log_P n) \\
&+ O\left(\frac{\ln n}{n}\right)
\end{aligned}
$$

where

$$
T_1^{*\prime}(0) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n!} \Gamma'(n) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n} \Psi(n) \ ,
$$

where $\Psi(z)$ *is the psi-function. The constant* $T_2^*(0)$ *can be computed as*

$$
T_2^*(0) = \sum_{n=2}^{\infty} \frac{w_n q^n}{n}
$$

where w_n *is given by the above recurrence. Finally,* $\delta_2(x)$ *is a periodic continuous function of zero mean and small amplitude. The constant* $[\delta_1^2]_0 = \sum_{k\neq 0} |\alpha_k|^2$ *is the zeroth term of* $\delta_1^2(x)$ *and its value is extremely small (e.g., for* $p = 0.5$ *one proves that* $[\delta_1^2]_0 \le \sup|\delta_1(x)|^2 \le 4 \times 10^{-10}$*).*

Numerical Evaluation

Table 2: Numerical evaluation of the constants $T_1^*(0)$, $T_1^{*\prime}(0)$, $T_2^*(0)$, and the variance $\mathrm{Var} H_n$ for various $p \in [0.2..0.8]$

p	$T_1^*(0)$	$T_1^{*\prime}(0)$	$T_2^*(0)$	$\mathrm{Var} H_n$
0.2	2.36	2.38	9.32	5.83
0.3	1.22	1.09	3.41	3.58
0.4	0.70	0.56	1.64	2.97
0.5	0.42	0.30	0.95	3.12
0.6	0.25	0.17	0.62	4.07
0.7	0.15	0.09	0.45	6.68
0.8	0.08	0.04	0.35	14.84

Limiting Distribution

Theorem 3. *The following holds, uniformly for all integers k,*

$$\mathbf{P}(H_n \le k) = F(p^k n) + O(n^{-1}), \qquad (1)$$

where

$$F(x) = x\int_0^\infty e^{-xt} d\mu(t) = \int_0^\infty e^{-t} d\mu_x(t), \qquad (2)$$

where the measure μ is defined on the positive real axis as follows:

Partition the positive real axis into an infinite sequence of consecutive intervals $I_0, I_1, \ldots$ such that I_k has length $(q/p)^{s(k)}$, where $s(k)$ is the number of 1's in the binary expansion of k. Thus, $I_0 = [0,1], I_1 = [1, 1 + q/p]$, etc. Note that the total length of the first 2^m intervals $I_0, \ldots, I_{2^m-1}$ is p^{-m}, and that these 2^m intervals are obtained by repeated subdivisions of $[0, p^{-m}]$, each time dividing each interval in the proportions $p : q$. Given these intervals, define μ by putting a point mass $|I_k|$ at the right endpoint of I_k, for each $k = 0, 1, \ldots$ with μ_x denoting the dilated measure defined as above for the intervals $xI_0, xI_1, \ldots$.

In particular, when $k = \lfloor \log_P n \rfloor + \kappa$ where κ is an integer, then for large n the following asymptotic formula is true uniformly over κ

$$\begin{aligned}\mathbf{P}(H_n \leq \lfloor \log_P n \rfloor + \kappa) &= p^{\kappa - \{\log_P n\}} \int_0^\infty e^{-tp^{\kappa - \{\log_P n\}}} d\mu(t) \\ &+ O\left(\frac{1}{n}\right),\end{aligned}$$

where $\{\log_P n\} = \log_P n - \lfloor \log_P n \rfloor$.

We observe that for the symmetric case $p = q = 0.5$ we obtain

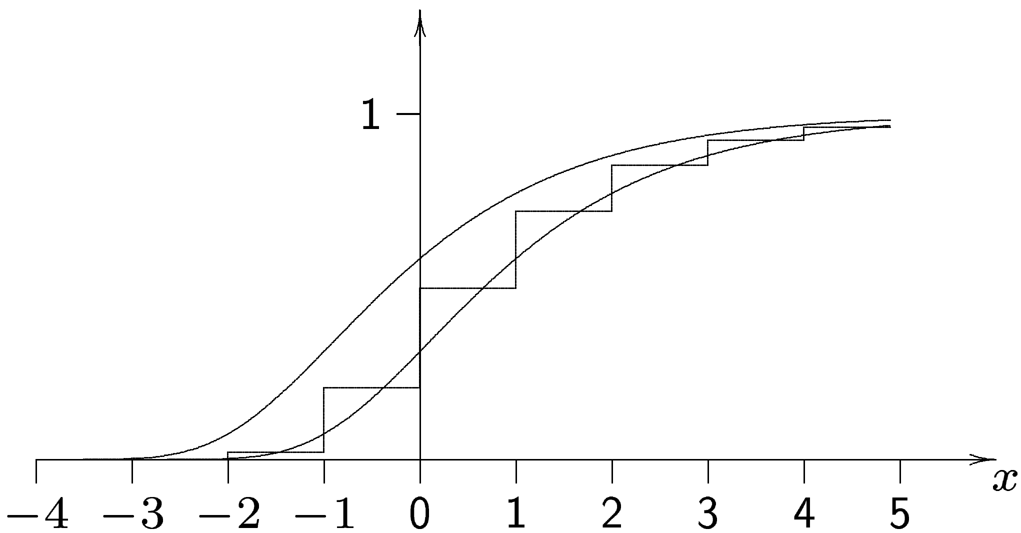
$$F(x) = x \sum_{j=1}^{\infty} e^{-jx} = \frac{x}{e^x - 1} ,$$

and our results coincide with those of Fill&Mahmoud&Szpankowski (1997):

$$\Pr\{H_n \leq \lfloor \lg n \rfloor + k\} = \frac{2^{\alpha(n)-k}}{\exp(2^{\alpha(n)-k}) - 1} + O\left(\frac{1}{\sqrt{n}}\right)$$

where $\alpha(n) := \lg n - \lfloor \lg n \rfloor$.

Distribution for Symmetric Leader Election



The distribution function of $H_{20} - \lfloor \lg 20 \rfloor$ and the two continuous extremes.

Sketch of Proof for DST

We only show how to estimate the Poisson mean $\widetilde{X}(z) = B'(1, z)$. Differentiating the basic functional equation one finds

$$(1 + \frac{\partial}{\partial z})^b \widetilde{X}(z) = z + \widetilde{X}(pz) + \widetilde{X}(qz)$$

This equation is amiable to the attack by Mellin transform.

To recall, for a function $f(x)$ or real valued x, we define its Mellin transform $F^*(s)$ as

$$F^*(s) = \mathcal{M}[f] = \int_0^\infty f(t) t^{s-1} dt \ .$$

One can find more on Mellin transform in an excellent survey by Flajolet, Gourdon and Dumas, *Theoretical Computer Science*, 144, 1995.

Useful Lemmas

Let $X(s)$ be the Mellin transform of $\widetilde{X}(z)$, and let

$$X(s) = \mathcal{M}[\widetilde{X}(t)] = \Gamma(s)\gamma(s)$$

for some function $\gamma(s)$.

Lemma 1. . *The following is true: $X(s)$ exists for $\Re(s) \in (-b-1, -1)$ is defined for $\Re(s) \in (-2b-1, -1)$. Furthermore, $\gamma(-1-i) = 0 \ for \ i = 1, ..., b-1, \ \gamma(-1-b) = (-1)^{b+1}$, and $\gamma(s)$ has simple poles at $s = -1, 0, 1, \ldots$.*

Lemma 2. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of real numbers, and suppose that its Poisson generating function $\widetilde{F}(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} e^{-z}$ is an entire function. Furthermore, let its Mellin transform $F(s)$ have the following factorization: $F(s) = \mathcal{M}[\widetilde{F}(z); s] = \Gamma(s)\gamma(s)$, and assume $F(s)$ exists for $\Re(s) \in (-2, -1)$ while $\gamma(s)$ is analytical for $\Re(s) \in (-\infty, -1)$. Then*

$$\gamma(-n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k, \quad \text{for} \quad n \geq 2.$$

Proof of Lemma 2

Let a sequence $\{g_n\}_{n=0}^{\infty}$ be such that $\widetilde{F}(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$, i.e.,

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_k \ , \qquad n \geq 0 \ .$$

If $\widetilde{F}_N(z) = \sum_{n=0}^{N-1} g_n \frac{z^n}{n!}$, then for $\Re(s) \in (-N, -N+1)$ we have

$$F(s) = \mathcal{M}[\widetilde{F}(z) - \widetilde{F}_N(z); s].$$

As $s \to -N$, due to the assumed factorization $F(s) = \Gamma(s)\gamma(s)$, we have

$$F(s) = \frac{1}{s+N} \frac{(-1)^N}{N!} \gamma(-N) + O(1) \ ;$$

and

$$\widetilde{F}(z) - \widetilde{F}_N(z) = \frac{(-1)^N}{N!} \gamma(-N) z^N + O(z^{N+1}) \quad \text{as} \quad z \to 0 \ .$$

Thus, $\gamma(-N) = (-1)^N g_N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k f_k$ for $N \geq 2$.

Proof

Then finding taking the Mellin transform of the basic functional equation on $\widetilde{X}(z)$, and using the above factorization, we obtain

$$\sum_{i=0}^{b} \binom{b}{i} (-1)^i \gamma(s-i) = (p^{-s} + q^{-s})\gamma(s) \ .$$

Define now

$$\widehat{\gamma}(s) = \sum_{i=1}^{b} \binom{b}{i} (-1)^{i+1} \gamma(s-i)$$

provided $\gamma(s-1), \ldots, \gamma(s-b)$ exist (at singularity points of $\gamma(s)$.)

Sketch of Proof for DST

Finally:

$$\begin{aligned}\gamma(s) &= \frac{1}{1-p^{-s}-q^{-s}}\sum_{i=1}^{b}\binom{b}{i}(-1)^{i+1}\gamma(s-i)\\ &= \frac{1}{1-p^{-s}-q^{-s}}\widehat{\gamma}(s) \ .\end{aligned}$$

Let now s_k, $k = 0, \pm 1, \pm 2, \ldots$ be roots of

$$1 - p^{-s} - q^{-s} = 0 \ .$$

Observe that $s_0 = -1$. Using

$$\frac{1}{1-p^{-s}-q^{-s}} = -\frac{1}{h(s_k)}\frac{1}{s-s_k} + \frac{h_2(s_k)}{2h^2(s_k)} + O(s-s_k)$$

we finally prove

$$X(s) = \Gamma(s)\gamma(s) = \frac{1}{h_1}\frac{1}{(s+1)^2} - \left(\frac{h_2}{2h_1^2} - \frac{\widehat{\gamma}'(-1) - \gamma + 1}{h_1}\right)\frac{1}{s+1}$$

where $\widehat{\gamma}'(-1)$ is a constant.

Sketch of Proof for DST

Using standard arguments of the inverse Mellin transform we finally arrive at

$$\begin{aligned}
\widetilde{X}(z) &= \frac{1}{h_1} z \log z + \left(\frac{h_2}{2h_1^2} - \frac{1}{h_1}\widehat{\gamma}'(-1) + \frac{\gamma - 1}{h_1} \right) z \\
&+ \sum_{k \neq 0} \frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h(s_k)} z^{-s_k} + O(1) \ .
\end{aligned}$$

The above asymptotic formula concerns the behavior of the Poisson mean as $z \to \infty$. Our original goal was to derive asymptotics of the mean ED_m in the Bernoulli model. To infer Bernoulli model behavior from it Poisson model asymptotics, we must apply the so called *depoissonization lemma*. Applying it we prove

$$\begin{aligned}
ED_m &= \widetilde{X}(m)/m \frac{1}{h_1} \log m + \frac{h_2}{2h_1^2} - \frac{1}{h_1}\widehat{\gamma}'(-1) + \frac{\gamma - 1}{h_1} \\
&+ \sum_{k \neq 0} \frac{\Gamma(s_k)\widehat{\gamma}(s_k)}{h(s_k)} m^{-1-s_k} + O\left(\frac{\log m}{m} \right) \ .
\end{aligned}$$

which proves Theorem 1.

Evaluation of the Constant

We must evaluate

$$\widehat{\gamma}'(-1) = \sum_{i=1}^{b} \binom{b}{i} (-1)^{i+1} \gamma'(-1-i) \ ,$$

where $\gamma(s)\Gamma(s) = \mathcal{M}[\widetilde{X}(t); s]$ and

$$\tilde{X}(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} e^{-z}.$$

We recall that

$$\begin{cases} f_{m+b} = m + \sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i} (f_i + f_{m-i}) \quad m \geq 0 \\ f_0 = f_1 = \ldots = f_b = 0, \\ \bar{f}_{m+b} = f_{m+b} - m \quad\quad m \geq 1 \ . \end{cases}$$

Evaluation of the Constant

Observe that for $b > 1$

$$\begin{aligned}\gamma(s) &= \frac{1}{\Gamma(s)}\mathcal{M}[\sum_{n=b+1}^{\infty} f_n \frac{z^n}{n!} e^{-z}; s] = \sum_{n=b+1}^{\infty} \frac{f_n}{n!}\frac{\Gamma(s+n)}{\Gamma(s)} \\ &= \sum_{n=b+1}^{\infty} \frac{f_n}{n!} s(s+1)...(s+n-1)\ .\end{aligned}$$

Thus for $\Re(s) \in (-b-1, -1)$:

$$\gamma'(s) = \sum_{n=b+1}^{\infty} \frac{f_n}{n!} s(s+1)\cdots(s+n-1) \sum_{i=0}^{n-1} \frac{1}{s+i}$$

for $s \notin \{-2, -3, \ldots, -b\}$.

Constant Evaluation

After some algebra:

$$\gamma'(-k) = -\frac{1}{b} - \frac{b}{b+1} + A + \Delta(b,p)$$

where

$$\begin{aligned}
\Delta(b,p) &= \sum_{n=b+2}^{\infty} \bar{f}_n \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\ldots(n-i-1)}\ , \\
A &= \sum_{n=b+2}^{\infty} (n-b) \sum_{i=1}^{b} \frac{(i+1)b!}{(b-i)!n(n-1)\ldots(n-i-1)}\ .
\end{aligned}$$

The above series converge since the summands are $O(\log n/n^2)$ for $b > 1$.

After tedious algebra, we can prove that

$$A = H_b + b(1+b)^{-1}$$

hence

$$\widehat{\gamma}'(-1) = H_{b-1} + \Delta(b,p)$$

as desired.

Sketch of Proof for LEA

The Poisson transform of x_n satisfies:

$$\widetilde{X}(z) = \widetilde{X}(pz) + \widetilde{X}(qz)e^{-pz} + 1 - (1+z)e^{-z}.$$

Define

$$T_1(z) = \widetilde{X}(qz)e^{-pz}$$

which is an entire function. Its Mellin $T_1^*(s)$ is defined in $\Re(s) \in (-2, \infty)$, hence the Mellin $X^*(s)$ of $\widetilde{X}(z)$ satisfy

$$X^*(s) = p^{-s}X^*(s) + T_1^*(s) - \Gamma(s) - \Gamma(s+1)$$

which has the following solution $(P = 1/p)$

$$X^*(s) = \frac{\Gamma(s) + \Gamma(s+1) - T_1^*(s)}{(P)^s - 1}, \qquad -1 < \Re s < 0.$$

Hence, the inverse Mellin gives us $\widetilde{X}(z)$ for large z, and depoissonization leads to x_n.

But the solution depends on $T_1^*(0)$, $T_2^*(0)$, and $T_1^{*\prime}(0)$.

Evaluation of the Constants

Since $\widetilde{X}(z) = \sum_{n\geq 2} x_n \frac{z^n}{n!} e^{-z}$ since $x_0 = x_1 = 0.$ and $\mathcal{M}(z^n e^{-z}, s) = \Gamma(s+n)$ for $\Re(s) > -n$, we obtain

$$X^*(s) = \sum_{n=2}^{\infty} \frac{x_n}{n!} \mathcal{M}(z^n e^{-z}, s) = \sum_{n=2}^{\infty} \frac{x_n}{n!} \Gamma(s+n)$$

provided $\Re(s) \in (-2, 0)$.

Moreover,

$$T_1(z) = \widetilde{X}(qz) e^{-pz} = \sum_{n\geq 2} x_n \frac{(qz)^n}{n!} e^{-z} = \sum_{n\geq 2} \frac{x_n q^n}{n!} z^n e^{-z}$$

and thus, similarly,

$$T_1^*(s) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n!} \Gamma(s+n)$$

provided $-2 < \Re(s) < \infty$. In particular,

$$T_1^*(0) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n!} \Gamma(n) = \sum_{n=2}^{\infty} \frac{x_n q^n}{n}.$$

Proof for Distribution

Let $\widetilde{G}_k(z) = \sum_{n=1}^{\infty} \mathbf{P}(H_n \leq k)\frac{z^n}{n!}e^{-z}$. Then

$$\begin{aligned}
\widetilde{G}_0(z) &= ze^{-z} \\
\widetilde{G}_{k+1}(z) &= \widetilde{G}_k(pz) + e^{-pz}\widetilde{G}_k(qz)\ , \qquad k \geq 0.
\end{aligned}$$

We claim that the above functional equation for $\widetilde{G}_k(z)$ is solved by

$$\widetilde{G}_k(z) = p^k z \int_0^{p^{-k}} e^{-p^k zt} d\mu(t).$$

Indeed, for $k \geq 1$, we use the fact that the measure μ on $(p^{-k}, p^{-k-1}]$ is obtained from μ on $(0, p^{-k}]$ by a translation and dilation, so that for every function f,

$$\int_{p^{-k}+}^{p^{-k-1}} f(t)d\mu(t) = \frac{q}{p}\int_0^{p^{-k}} f(p^{-k} + \frac{q}{p}t)d\mu(t)$$

and thus

$$\int_0^{p^{-k-1}} f(t)d\mu(t) = \int_0^{p^{-k}} f(t)d\mu(t) + \frac{q}{p}\int_0^{p^{-k}} f(p^{-k} + \frac{q}{p}t)d\mu(t).$$