MONODROMY OF GENERALIZED POLYLOGARITHMS

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Joint work: with Joris Van Der Hoeven and Michel Petitot (Lille I).

Séminaire Algorithmes, Rocquencourt, 6 Juillet 1998.

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POLYLOGARITHMS AND COMBINATORICS ON WORDS

Definitions

Let $X = \{x_0, x_1\}$. To any word $w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1 \dots x_0^{s_k-1}x_1 \in X^*x_1$, we associate the multi–index $s = (s_1, s_2, \dots, s_k)$.

Definition 1 *Polylogarithms (Generalized polylogarithms):*

$$Li_w(z) = Li_s(z) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}.$$

Definition 2 *Multiple Zeta Values (MZV):*

$$\zeta_w = \zeta(s) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}.$$

Lyndon words and Radford's theorem

A *Lyndon word* is a non empty word which is inferior to each of its strict right factors (for the lexicographical ordering).
The set of Lyndon words is denoted by $\mathcal{L}yn(X)$.

EXAMPLE – For $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length $\leqslant 6$ on X^* are the following (in lexicographical increasing order)

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2,$$
$$x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

□

Theorem 1 (Radford) *The* $\mathbb{Q}$*–algebra* $\mathrm{Sh}_{\mathbb{Q}}\langle X\rangle$ *is the algebra of polynomials generated by the Lyndon words.*

Bracket forms and the dual basis

The *bracket form* $P(l)$ of a Lyndon word l is defined recursively by

$$\left\{ \begin{array}{rcl} P(l) & = & [P(u), P(v)] \quad \text{for } uv = \mathrm{st}(l), \\ P(x) & = & x \quad \text{for each letter } x \in X, \end{array} \right.$$

The set $\{P(l);\ l \in \mathcal{L}yn(X)\}$ is a basis for the free Lie algebra.

The PBW basis $\mathcal{B} = \{P(w);\ w \in X^*\}$ and its dual basis $\mathcal{B}^* = \{P^*(w);\ w \in X^*\}$ are obtained from by setting

$$\left\{ \begin{array}{rcll} P(w) & = & P(l_1)^{i_1} \dots P(l_k)^{i_k}, & w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, \\ P^*(w) & = & \dfrac{P^*(l_1)^{\text{ш}\, i_1} \,\text{ш}\, \dots \,\text{ш}\, P^*(l_k)^{\text{ш}\, i_k}}{i_1! \dots i_k!}, & w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, \\ P^*(l) & = & xP^*(w), \quad \forall l \in \mathcal{L}yn(X), & l = xw,\ x \in X,\ w \in X^*. \end{array} \right.$$

Lie exponentials

A series $S \in R\langle\langle X \rangle\rangle$ is called a *Lie exponential* if there exists a *Lie series* $L \in \mathcal{L}ie_R\langle\langle X \rangle\rangle$ such that $S = e^L$. This is equivalent to

$$\forall u, v \in X^*, \ (S|u \sqcup\!\sqcup v) = (S|u)(S|v).$$

The product of two Lie exponentials is a Lie exponential.

Let $S \in R\langle\langle X \rangle\rangle$ be a Lie exponential. Then S can be factored as an infinite product of Lie exponentials

$$S = \sum_{w \in X^*} (S|w) \ w = \prod_{l \in \mathcal{L}yn(X)\searrow} e^{(S|P^*(l))P(l)}.$$

Examples

l	$P(l)$	$P^*(l)$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0,\ x_1]$	x_0x_1
${x_0}^2x_1$	$[x_0,\ [x_0,\ x_1]]$	${x_0}^2x_1$
$x_0{x_1}^2$	$[[x_0,\ x_1],\ x_1]$	$x_0{x_1}^2$
${x_0}^3x_1$	$[x_0,\ [x_0,\ [x_0,\ x_1]]]$	${x_0}^3x_1$
$\cdots$	$\cdots$	$\cdots$
${x_0}^3{x_1}^3$	$[x_0,\ [x_0,\ [[[x_0,\ x_1],\ x_1],\ x_1]]]$	${x_0}^3{x_1}^3$
${x_0}^2x_1x_0{x_1}^2$	$[x_0,\ [[x_0,\ x_1],\ [[x_0,\ x_1],\ x_1]]]$	$3{x_0}^3{x_1}^3 + {x_0}^2x_1x_0{x_1}^2$
${x_0}^2{x_1}^2x_0x_1$	$[[x_0,\ [[x_0,\ x_1],\ x_1]],\ [x_0,\ x_1]]$	$6{x_0}^3{x_1}^3 + 3{x_0}^2x_1x_0{x_1}^2 + {x_0}^2{x_1}^2x_0x_1$
${x_0}^2{x_1}^4$	$[x_0,\ [[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1]]$	${x_0}^2{x_1}^4$
$x_0x_1x_0{x_1}^3$	$[[x_0,\ x_1],\ [[[x_0,\ x_1],\ x_1],\ x_1]]$	$4{x_0}^2{x_1}^4 + x_0x_1x_0{x_1}^3$
$x_0{x_1}^5$	$[[[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1],\ x_1]$	$x_0{x_1}^5$

RELATIONS BETWEEN MULTIPLE ZETA VALUES

Table

$$
\begin{aligned}
\zeta(2,\,1) &= \zeta(3) \\
\zeta(4) &= \frac{2}{5}\,\zeta(2)^2 \\
\zeta(3,\,1) &= \frac{1}{10}\,\zeta(2)^2 \\
\zeta(2,\,1,\,1) &= \frac{2}{5}\,\zeta(2)^2 \\
\zeta(4,\,1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3) \\
\zeta(3,\,2) &= -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3) \\
\zeta(3,\,1,\,1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3) \\
\zeta(2,\,2,\,1) &= -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3) \\
\zeta(2,\,1,\,1,\,1) &= \zeta(5) \\
&\vdots
\end{aligned}
$$

The Idea

1. There existes two shuffle structures for Multiple Zeta Values.
2. The collision of these two structures gives Grobner basis for the ideal of the relations.
3. Pure symbolic computation.

First shuffle product

Let $X = \{x_0, x_1\}$.

$$w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1\cdots x_0^{s_k-1}x_1 \longrightarrow \zeta_w$$
$$p \in \mathrm{Sh}_{\mathbb{Q}}\langle X\rangle \longrightarrow \zeta_p \in \mathbb{R} \quad \text{by linearity}$$

Theorem 2 *From the theorie of* iterated integrales, *we have*

$$\forall u, v \in x_0X^*x_1, \quad \zeta_{u \sqcup\!\sqcup v} = \zeta_u\zeta_v.$$

Proof – Use the fact that:

$$Li_\varepsilon(z) = 1, \quad Li_{x_0w}(z) = \int_0^z \frac{dt}{t}Li_w(t), \quad Li_{x_1w}(z) = \int_0^z \frac{dt}{1-t}Li_w(t).$$

□

Example –

$$x_0x_1 \sqcup\!\sqcup x_0x_1 = 4x_0x_0x_1x_1 + 2x_0x_1x_0x_1$$
$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2)$$

□

Second shuffle product

Let $Y = \{y_1, y_2, y_3, \ldots\}$. $\qquad y_i \longleftrightarrow x_0^{i-1}x_1.$

$$w = y_{s_1}y_{s_2}\ldots y_{s_k} \longrightarrow \zeta_w$$
$$p \in \mathbb{Q}\langle Y\rangle \longrightarrow \zeta_p \in \mathbb{R} \quad \text{by linearity}$$

Definition 3

$$u * v = y_i(u' * v) + y_j(u * v') + y_{i+j}(u' * v') \quad \textit{for } u = y_iu',\ v = y_jv'$$

Theorem 3 *From the theorie of* quasi–symetrique functions, *we have*

$$\forall u, v \in Y^*, \quad \zeta_{u*v} = \zeta_u\zeta_v.$$

Example –

$$y_2 * y_3y_1 = y_2y_3y_1 + y_3y_2y_1 + y_1y_3y_2 + y_5y_1 + y_3y_3$$
$$\zeta(2)\zeta(3,1) = \zeta(2,3,1) + \zeta(3,2,1) + \zeta(1,3,2) + \zeta(5,1) + \zeta(3,3)$$

□

Generation of a Groebner basis of relations

$$\begin{cases} \forall l_1, \in \mathcal{L}yn(X), |l_1| \geq 2, & \zeta_{l_1 \sqcup\!\sqcup x_1} - \zeta_{l_1 * x_1} = \zeta_{l_1 \sqcup\!\sqcup x_1 - l_1 * x_1} = 0, \\ \forall l_1, l_2 \in \mathcal{L}yn(X), |l_1| \geq 2, |l_2| \geq 2, & \zeta_{l_1 \sqcup\!\sqcup l_2} - \zeta_{l_1 * l_2} = \zeta_{l_1 \sqcup\!\sqcup l_2 - l_1 * l_2} = 0. \end{cases}$$

i.e.

$$\begin{cases} \forall l_1, \in \mathcal{L}yn(X), |l_1| \geq 2, & l_1 \sqcup\!\sqcup x_1 - l_1 * x_1 \in Ker(\zeta), \\ \forall l_1, l_2 \in \mathcal{L}yn(X), |l_1| \geq 2, |l_2| \geq 2, & l_1 \sqcup\!\sqcup l_2 - l_1 * l_2 \in Ker(\zeta). \end{cases}$$

Methode: decomposition of $\left\{ \begin{array}{l} l_1 \sqcup\!\sqcup x_1 - l_1 * x_1 \\ l_1 \sqcup\!\sqcup l_2 - l_1 * l_2 \end{array} \right\}$ as shuffle of Lyndon words. The Lyndon words are considered as new commutative variables.

The Zagier's dimension conjecture

Conjecture 1 *Let d_n be a dimension of the $\mathbb{Q}$-vector space generated by $\zeta(s_1, \ldots, s_k) = s_1 + \ldots + s_k$. Then*

$$d_1 = 0, \quad d_2 = d_3 = 1, \quad d_n = d_{n-2} + d_{n-3}, \qquad n \geq 4.$$

We can then conjecture:

Conjecture 2 *The $\mathbb{Q}$-algebra of the $\zeta(s)$ is a polynomial algebra.*

It is checked up to order 10 if one admits Zagier's dimension conjecture:

$$\zeta(2), \zeta(3), \zeta(5), \zeta(7), \zeta(9), \zeta(6,2), \zeta(8,2)$$

are free.

MONODROMY OF POLYLOGARITHMS

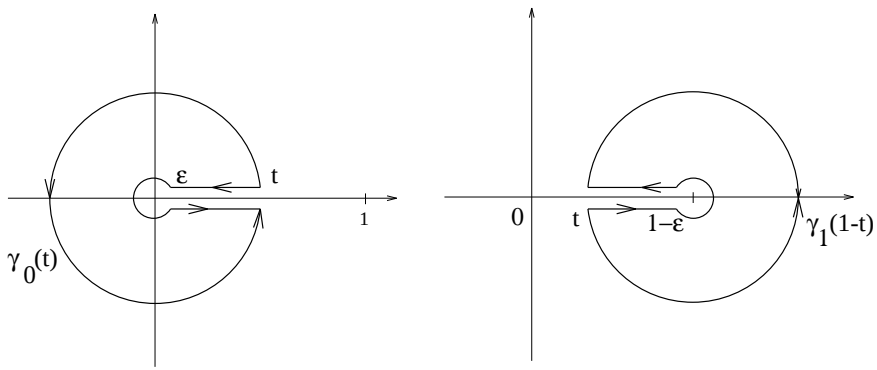

Figure 1: Paths of integration

Monodromy of polylogarithms

We prove the monodromy of the polylogarithms Li_w is given by

$$\begin{aligned} \mathcal{M}_0 Li_{wx_0} &= Li_{wx_0} + 2i\pi Li_w + \cdots \\ \mathcal{M}_1 Li_{wx_1} &= Li_{wx_1} - 2i\pi Li_w + \cdots , \end{aligned}$$

where the remaining terms are linear combinations of polylogarithms coded by words of lengths $< |w|$.

Monodromy arround $z = 1$ by AXIOM

$$p = 2i\pi$$

$$
\begin{aligned}
\mathcal{M}_1 Li_{x_0} &= Li_{x_0} \\
\mathcal{M}_1 Li_{x_1} &= Li_{x_1} - p \\
\mathcal{M}_1 Li_{x_0 x_1} &= Li_{x_0 x_1} - p Li_{x_0} \\
\mathcal{M}_1 Li_{x_0^2 x_1} &= Li_{x_0^2 x_1} - \tfrac{1}{2} p {Li_{x_0}}^2 \\
\mathcal{M}_1 Li_{x_0 x_1^2} &= Li_{x_0 x_1^2} - p Li_{x_0 x_1} + \tfrac{1}{2} p^2 Li_{x_0} + p\zeta_{x_0 x_1} \\
\mathcal{M}_1 Li_{x_0^3 x_1} &= Li_{x_0^3 x_1} - \tfrac{1}{6} p {Li_{x_0}}^3 \\
\mathcal{M}_1 Li_{x_0^2 x_1^2} &= Li_{x_0^2 x_1^2} - p Li_{x_0^2 x_1} + \tfrac{1}{4} p^2 {Li_{x_0}}^2 + p\zeta_{x_0 x_1} Li_{x_0} + p\zeta_{x_0^2 x_1} \\
\mathcal{M}_1 Li_{x_0 x_1^3} &= Li_{x_0 x_1^3} - p Li_{x_0 x_1^2} + \tfrac{1}{2} p^2 Li_{x_0 x_1} - \tfrac{1}{6} p^3 Li_{x_0} + p\zeta_{x_0 x_1^2} - \tfrac{1}{2} p^2 \zeta_{x_0 x_1} \\
\mathcal{M}_1 Li_{x_0^4 x_1} &= Li_{x_0^4 x_1} - \tfrac{1}{24} p {Li_{x_0}}^4 \\
&\vdots
\end{aligned}
$$

Non commutative generating series of polylogarithms

Definition 4 *The generating series L of polylogarithms is:*

$$L(z) = \sum_{w \in X^*} Li_w(z) w.$$

It is useful to extend the definition of Li_w over X^* : $Li_{x_0^n}(z) = \log^n(z)/n!$.

L satisfies the differential equation with border condition

$$\begin{aligned} \frac{d}{dz} L(z) &= \left(\frac{x_0}{z} + \frac{x_1}{1-z} \right) L(z) \\ L(\varepsilon) &= e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon}) \quad \text{where } \varepsilon \to 0^+. \end{aligned}$$

Monodromy of L

Theorem 4 *The monodromy of the series $L(t)$ for $t \in]0,1[$ around $z=0$ and $z=1$ is given by*

$$\begin{cases} \mathcal{M}_0 L(t) &= L(t) e^{2i\pi \mathfrak{m}_0}, \\ \mathcal{M}_1 L(t) &= L(t) e^{2i\pi \mathfrak{m}_1}, \end{cases}$$

where

$$\mathfrak{m}_0 = x_0$$

and $\mathfrak{m}_1$ is a Lie serie *given by the formula*

$$\mathfrak{m}_1 = \prod_{l \notin \{x_0, x_1\}^{\nearrow}} e^{-\zeta_{P^*(l)} \operatorname{ad} P(l)}(-x_1).$$

The series $\mathfrak{m}_1$ up to order 6 by AXIOM

$$
\begin{aligned}
\mathfrak{m}_1 \;=\; & -[x_1] + \zeta_{x_0x_1}\left[x_0{x_1}^2\right] + \zeta_{{x_0}^2x_1}\left[{x_0}^2{x_1}^2\right] + \zeta_{x_0{x_1}^2}\left[x_0{x_1}^3\right] + \zeta_{{x_0}^3x_1}\left[{x_0}^3{x_1}^2\right] \\
& - \zeta_{{x_0}^3x_1}\left[{x_0}^2x_1x_0x_1\right] + \zeta_{{x_0}^2{x_1}^2}\left[{x_0}^2{x_1}^3\right] + \left(\zeta_{{x_0}^2{x_1}^2} - \frac{1}{2}{\zeta_{x_0x_1}}^2\right)\left[x_0x_1x_0{x_1}^2\right] \\
& + \zeta_{x_0{x_1}^3}\left[x_0{x_1}^4\right] + \zeta_{{x_0}^4x_1}\left[{x_0}^4{x_1}^2\right] - 2\zeta_{{x_0}^4x_1}\left[{x_0}^3x_1x_0x_1\right] \\
& + \zeta_{{x_0}^3{x_1}^2}\left[{x_0}^3{x_1}^3\right] + \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2x_1x_0{x_1}^2\right] \\
& + \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{x_0x_1}\zeta_{{x_0}^2x_1} + 2\zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2{x_1}^2x_0x_1\right] \\
& + \zeta_{{x_0}^2{x_1}^3}\left[{x_0}^2{x_1}^4\right] + \left(4\zeta_{{x_0}^2{x_1}^3} + \zeta_{x_0x_1x_0{x_1}^2}\right)\left[x_0x_1x_0{x_1}^3\right] + \zeta_{x_0{x_1}^4}\left[x_0{x_1}^5\right]
\end{aligned}
$$

INDEPENDENCE OF POLYLOGARITHMS

A structure theorem

Theorem 5 *The functions* $\{Li_w\}_{w\in X^*}$ *are* $\mathbb{C}$*-linearly independent.*

$$w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1\cdots x_0^{s_k-1}x_1 \longrightarrow Li_w$$
$$p \in \mathrm{Sh}_{\mathbb{C}}\langle X\rangle \longrightarrow Li_p \quad \text{by linearity}$$

Corollary 1 *The* $\mathbb{C}$*-algebra of polylogarithms is isomorphic to the shuffle algebra.*

Corollary 2 *The polylogarithms coded by the words* $l \in \mathcal{L}yn(X)$ *(resp. the polynomials* $P^*(l), l \in \mathcal{L}yn(X)$*) form an infinite transcendence basis.*

A induction proof of the structure theorem

- This is trivial for $n = 0$.
- Assume that we proved our assertion for all $k, 0 \leq k \leq n-1$.
- For $k = n$ (the λ_w are elements of $\mathbb{C}$)

$$\sum_{|w|\leqslant n} \lambda_w Li_w = 0 \iff \lambda_1 + \sum_{|u|<n} \lambda_{ux_0} Li_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} Li_{ux_1} = 0.$$

Applying the operators $(\mathcal{M}_0 - Id)$ and $(Id - \mathcal{M}_1)$, we obtain

$$\begin{cases} 2i\pi \displaystyle\sum_{|u|=n-1} \lambda_{ux_0} Li_u + \sum_{|u|<n-1} \mu_u Li_u = 0, \\ 2i\pi \displaystyle\sum_{|u|=n-1} \lambda_{ux_1} Li_u + \sum_{|u|<n-1} \nu_u Li_u = 0. \end{cases}$$

By the induction hypothesis, we get the expected result.