MONODROMY OF GENERALIZED POLYLOGAR

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POLYLOGARITHMS AND COMBINATORICS ON

Definitions

Let $X = \{x_0, x_1\}$. To any word $w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1 \ldots x_0^{s_k-1}x_1$ we associate the multi–index $s = (s_1, s_2, \ldots, s_k)$.

Definition 1 *Polylogarithms (Generalized polylogarithms):*

$$Li_w(z) = Li_s(z) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}.$$

Definition 2 *Multiple Zeta Values (MZV):*

$$\zeta_w = \zeta(s) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}.$$

Lyndon words and Radford's theorem

A *Lyndon word* is a non empty word which is inferior to each o
right factors (for the lexicographical ordering).
The set of Lyndon words is denoted by $\mathcal{L}yn(X)$.

EXAMPLE – For $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon word
length $\leqslant 6$ on X^* are the following (in lexicographical increasin

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, \\ x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

□

Theorem 1 (Radford) *The* $\mathbb{Q}$*–algebra* $\mathrm{Sh}_{\mathbb{Q}}\langle X\rangle$ *is the algebr polynomials generated by the Lyndon words.*

Bracket forms and the dual basis

The *bracket form* $P(l)$ of a Lyndon word l is defined recursively

$$\begin{cases} P(l) = [P(u), P(v)] & \text{for } uv = \mathrm{st}(l), \\ P(x) = x & \text{for each letter } x \in X, \end{cases}$$

The set $\{P(l);\ l \in \mathcal{L}yn(X)\}$ is a basis for the free Lie algebra.

The PBW basis $\mathcal{B} = \{P(w);\ w \in X^*\}$ and its dual basis $\mathcal{B}^* = \{P^*(w);\ w \in X^*\}$ are obtained from by setting

$$\begin{cases} P(w) = P(l_1)^{i_1} \ldots P(l_k)^{i_k}, & w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots \\ P^*(w) = \dfrac{P^*(l_1)^{ш\, i_1} ш \ldots ш P^*(l_k)^{ш\, i_k}}{i_1! \ldots i_k!}, & w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots \\ P^*(l) = xP^*(w), \quad \forall l \in \mathcal{L}yn(X), & l = xw,\ x \in X,\ w \end{cases}$$

Lie exponentials

A series $S \in R\langle\langle X\rangle\rangle$ is called a *Lie exponential* if there exists a *series* $L \in \mathcal{L}ie_R\langle\langle X\rangle\rangle$ such that $S = e^L$. This is equivalent to

$$\forall u, v \in X^*,\ (S|u \sqcup\!\sqcup v) = (S|u)(S|v).$$

The product of two Lie exponentials is a Lie exponential.

Let $S \in R\langle\langle X\rangle\rangle$ be a Lie exponential. Then S can be factored a infinite product of Lie exponentials

$$S = \sum_{w \in X^*} (S|w)\ w = \prod_{l \in \mathcal{L}yn(X)\searrow} e^{(S|P^*(l))P(l)}.$$

Examples

l	$P(l)$	$P^*(l)$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0,\ x_1]$	x_0x_1
${x_0}^2x_1$	$[x_0,\ [x_0,\ x_1]]$	${x_0}^2x_1$
$x_0{x_1}^2$	$[[x_0,\ x_1],\ x_1]$	$x_0{x_1}^2$
${x_0}^3x_1$	$[x_0,\ [x_0,\ [x_0,\ x_1]]]$	${x_0}^3x_1$
$\ldots$	$\ldots$	$\ldots$
${x_0}^3{x_1}^3$	$[x_0,\ [x_0,\ [[[x_0,\ x_1],\ x_1],\ x_1]]]$	${x_0}^3{x_1}^3$
${x_0}^2x_1x_0{x_1}^2$	$[x_0,\ [[x_0,\ x_1],\ [[x_0,\ x_1],\ x_1]]]$	$3{x_0}^3{x_1}^3 + {x_0}^2x_1x_0{x_1}^2$
${x_0}^2{x_1}^2x_0x_1$	$[[x_0,\ [[x_0,\ x_1],\ x_1]],\ [x_0,\ x_1]]$	$6{x_0}^3{x_1}^3 + 3{x_0}^2x_1x_0{x_1}^2 + {x_0}^2x$
${x_0}^2{x_1}^4$	$[x_0,\ [[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1]]$	${x_0}^2{x_1}^4$
$x_0x_1x_0{x_1}^3$	$[[x_0,\ x_1],\ [[[x_0,\ x_1],\ x_1],\ x_1]]$	$4{x_0}^2{x_1}^4 + x_0x_1x_0{x_1}^3$
$x_0{x_1}^5$	$[[[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1],\ x_1]$	$x_0{x_1}^5$

RELATIONS BETWEEN MULTIPLE ZETA VAL

Table

$$
\begin{aligned}
\zeta(2,\ 1) &= \zeta(3)\\
\zeta(4) &= \frac{2}{5}\,\zeta(2)^2\\
\zeta(3,\ 1) &= \frac{1}{10}\,\zeta(2)^2\\
\zeta(2,\ 1,\ 1) &= \frac{2}{5}\,\zeta(2)^2\\
\zeta(4,\ 1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3)\\
\zeta(3,\ 2) &= -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3)\\
\zeta(3,\ 1,\ 1) &= 2\,\zeta(5) - \zeta(2)\,\zeta(3)\\
\zeta(2,\ 2,\ 1) &= -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3)\\
\zeta(2,\ 1,\ 1,\ 1) &= \zeta(5)\\
&\vdots
\end{aligned}
$$

The Idea

1. There existes two shuffle structures for Multiple Zeta Values
2. The collision of these two structures gives Grobner basis for of the relations.
3. Pure symbolic computation.

First shuffle product

Let $X = \{x_0, x_1\}$.

$$w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1 \cdots x_0^{s_k-1}x_1 \longrightarrow \zeta_w$$
$$p \in \mathrm{Sh}_{\mathbb{Q}}\langle X\rangle \longrightarrow \zeta_p \in \mathbb{R} \quad \text{by linearit}$$

Theorem 2 *From the theorie of* iterated integrales, *we have*

$$\forall u, v \in x_0X^*x_1, \quad \zeta_{u \sqcup\!\sqcup v} = \zeta_u\zeta_v.$$

Proof – Use the fact that:

$$Li_\varepsilon(z) = 1, \quad Li_{x_0w}(z) = \int_0^z \frac{dt}{t}Li_w(t), \quad Li_{x_1w}(z) = \int_0^z \frac{dt}{1-t$$

□

Example –

$$x_0x_1 \sqcup\!\sqcup x_0x_1 = 4x_0x_0x_1x_1 + 2x_0x_1x_0x_1$$
$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2)$$

□

Second shuffle product

Let $Y = \{y_1, y_2, y_3, \ldots\}$. $y_i \longleftarrow$

$$w = y_{s_1} y_{s_2} \ldots y_{s_k} \longrightarrow \zeta_w$$
$$p \in \mathbb{Q}\langle Y \rangle \longrightarrow \zeta_p \in \mathbb{R} \quad \text{by linearity}$$

Definition 3

$$u * v = y_i(u' * v) + y_j(u * v') + y_{i+j}(u' * v') \quad \textit{for } u = y_i u', \ v$$

Theorem 3 *From the theorie of* quasi–symetrique functions, u

$$\forall u, v \in Y^*, \quad \zeta_{u*v} = \zeta_u \zeta_v.$$

Example –

$$y_2 * y_3 y_1 = y_2 y_3 y_1 + y_3 y_2 y_1 + y_1 y_3 y_2 + y_5 y_1 + y_3 y_3$$
$$\zeta(2)\zeta(3,1) = \zeta(2,3,1) + \zeta(3,2,1) + \zeta(1,3,2) + \zeta(5,1) + \zeta$$

□

Generation of a Groebner basis of relations

$$\begin{cases} \forall l_1, \in \mathcal{L}yn(X), |l_1| \geq 2, & \zeta_{l_1 ш x_1} - \zeta_{l_1 * x_1} = \zeta_{l_1 ш x_1 -} \\ \forall l_1, l_2 \in \mathcal{L}yn(X), |l_1| \geq 2, |l_2| \geq 2, & \zeta_{l_1 ш l_2} - \zeta_{l_1 * l_2} = \zeta_{l_1 ш l_2 -} \end{cases}$$

i.e.

$$\begin{cases} \forall l_1, \in \mathcal{L}yn(X), |l_1| \geq 2, & l_1 ш x_1 - l_1 * x_1 \in Ke \\ \forall l_1, l_2 \in \mathcal{L}yn(X), |l_1| \geq 2, |l_2| \geq 2, & l_1 ш l_2 - l_1 * l_2 \in Ke \end{cases}$$

Methode: decomposition of $\left\{ \begin{array}{c} l_1 ш x_1 - l_1 * x_1 \\ l_1 ш l_2 - l_1 * l_2 \end{array} \right\}$ as shuffle of Ly words. The Lyndon words are considered as new commutative v

The Zagier's dimension conjecture

Conjecture 1 *Let d_n be a dimension of the $\mathbb{Q}$-vector space g* *by $\zeta(s_1, \ldots, s_k) = s_1 + \ldots + s_k$. Then*

$$d_1 = 0, \quad d_2 = d_3 = 1, \quad d_n = d_{n-2} + d_{n-3}, \qquad n \geq 4$$

We can then conjecture:

Conjecture 2 *The $\mathbb{Q}$-algebra of the $\zeta(s)$ is a polynomial alge*

It is checked up to order 10 if one admits Zagier's dimension con

$$\zeta(2), \zeta(3), \zeta(5), \zeta(7), \zeta(9), \zeta(6,2), \zeta(8,2)$$

are free.

MONODROMY OF POLYLOGARITHMS

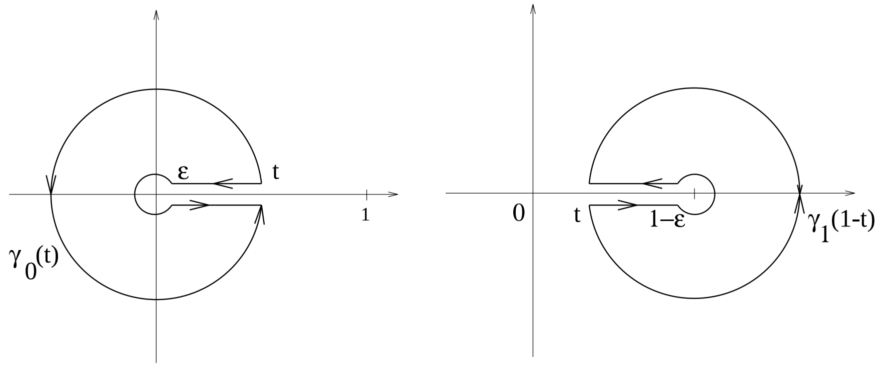

Figure 1: Paths of integration

Monodromy of polylogarithms

We prove the monodromy of the polylogarithms Li_w is given by

$$\begin{aligned} \mathcal{M}_0 Li_{wx_0} &= Li_{wx_0} + 2i\pi Li_w + \cdots \\ \mathcal{M}_1 Li_{wx_1} &= Li_{wx_1} - 2i\pi Li_w + \cdots , \end{aligned}$$

where the remaining terms are linear combinations of polylogari
coded by words of lengths $< |w|$.

Monodromy arround $z = 1$ by AXIOM

$$p = 2i\pi$$

$$\begin{aligned}
\mathcal{M}_1 Li_{x_0} &= Li_{x_0} \\
\mathcal{M}_1 Li_{x_1} &= Li_{x_1} - p \\
\mathcal{M}_1 Li_{x_0x_1} &= Li_{x_0x_1} - pLi_{x_0} \\
\mathcal{M}_1 Li_{x_0^2x_1} &= Li_{x_0^2x_1} - \tfrac{1}{2}pLi_{x_0}{}^2 \\
\mathcal{M}_1 Li_{x_0x_1^2} &= Li_{x_0x_1^2} - pLi_{x_0x_1} + \tfrac{1}{2}p^2Li_{x_0} + p\zeta_{x_0x_1} \\
\mathcal{M}_1 Li_{x_0^3x_1} &= Li_{x_0^3x_1} - \tfrac{1}{6}pLi_{x_0}{}^3 \\
\mathcal{M}_1 Li_{x_0^2x_1^2} &= Li_{x_0^2x_1^2} - pLi_{x_0^2x_1} + \tfrac{1}{4}p^2Li_{x_0}{}^2 + p\zeta_{x_0x_1}Li_{x_0} + p\zeta_{x_0^2x_1} \\
\mathcal{M}_1 Li_{x_0x_1^3} &= Li_{x_0x_1^3} - pLi_{x_0x_1^2} + \tfrac{1}{2}p^2Li_{x_0x_1} - \tfrac{1}{6}p^3Li_{x_0} + p\zeta_{x_0x_1^2} - \\
\mathcal{M}_1 Li_{x_0^4x_1} &= Li_{x_0^4x_1} - \tfrac{1}{24}pLi_{x_0}{}^4 \\
&\vdots
\end{aligned}$$

Non commutative generating series of polylogarith

Definition 4 *The generating series L of polylogarithms is:*

$$L(z) = \sum_{w \in X^*} Li_w(z) w.$$

It is useful to extend the definition of Li_w over X^* : $Li_{x_0^n}(z) = $ lo

L satisfies the differential equation with border condition

$$\frac{d}{dz} L(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z} \right) L(z)$$
$$L(\varepsilon) = e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon}) \quad \text{where } \varepsilon \to 0^+.$$

Monodromy of L

Theorem 4 *The monodromy of the series $L(t)$ for $t \in]0,1[$ a*
$z=0$ and $z=1$ is given by

$$\begin{cases} \mathcal{M}_0 L(t) &= L(t)e^{2i\pi\mathfrak{m}_0}, \\ \mathcal{M}_1 L(t) &= L(t)e^{2i\pi\mathfrak{m}_1}, \end{cases}$$

where

$$\mathfrak{m}_0 = x_0$$

and $\mathfrak{m}_1$ is a Lie serie *given by the formula*

$$\mathfrak{m}_1 = \prod_{l \notin \{x_0,x_1\} \nearrow} e^{-\zeta_{P^*(l)} \operatorname{ad} P(l)}(-x_1).$$

The series $\mathfrak{m}_1$ up to order 6 by Axiom

$$
\begin{aligned}
\mathfrak{m}_1 \;=\; & -[x_1] + \zeta_{x_0x_1}\left[x_0{x_1}^2\right] + \zeta_{{x_0}^2x_1}\left[{x_0}^2{x_1}^2\right] + \zeta_{x_0{x_1}^2}\left[x_0{x_1}^3\right] + \zeta_{x_0}\cdots \\
- \;& \zeta_{{x_0}^3x_1}\left[{x_0}^2x_1x_0x_1\right] + \zeta_{{x_0}^2{x_1}^2}\left[{x_0}^2{x_1}^3\right] + \left(\zeta_{{x_0}^2{x_1}^2} - \frac{1}{2}{\zeta_{x_0x_1}}^2\right)\cdots \\
+ \;& \zeta_{x_0{x_1}^3}\left[x_0{x_1}^4\right] + \zeta_{{x_0}^4x_1}\left[{x_0}^4{x_1}^2\right] - 2\zeta_{{x_0}^4x_1}\left[{x_0}^3x_1x_0x_1\right] \\
+ \;& \zeta_{{x_0}^3{x_1}^2}\left[{x_0}^3{x_1}^3\right] + \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2x_1x_0{x_1}^2\right] \\
+ \;& \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{x_0x_1}\zeta_{{x_0}^2x_1} + 2\zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2{x_1}^2x_0x_1\right] \\
+ \;& \zeta_{{x_0}^2{x_1}^3}\left[{x_0}^2{x_1}^4\right] + \left(4\zeta_{{x_0}^2{x_1}^3} + \zeta_{x_0x_1x_0{x_1}^2}\right)\left[x_0x_1x_0{x_1}^3\right] + \zeta_{x_0x_1}\cdots
\end{aligned}
$$

INDEPENDENCE OF POLYLOGARITHMS

A structure theorem

Theorem 5 *The functions $\{Li_w\}_{w \in X^*}$ are $\mathbb{C}$-linearly indepen*

$$w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1\cdots x_0^{s_k-1}x_1 \longrightarrow Li_w$$
$$p \in \mathrm{Sh}_{\mathbb{C}}\langle X\rangle \longrightarrow Li_p \quad \text{by linearity}$$

Corollary 1 *The $\mathbb{C}$-algebra of polylogarithms is isomorphic t shuffle algebra.*

Corollary 2 *The polylogarithms coded by the words $l \in \mathcal{L}yn($ the polynomials $P^*(l), l \in \mathcal{L}yn(X)$) form an infinite transcend*

A induction proof of the structure theorem

- This is trivial for $n = 0$.
- Assume that we proved our assertion for all $k, 0 \leq k \leq n-1$.
- For $k = n$ (the λ_w are elements of $\mathbb{C}$)

$$\sum_{|w|\leqslant n} \lambda_w Li_w = 0 \iff \lambda_1 + \sum_{|u|<n} \lambda_{ux_0} Li_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} Li_{u}$$

Applying the operators $(\mathcal{M}_0 - Id)$ and $(Id - \mathcal{M}_1)$, we obtain

$$\begin{cases} 2i\pi \sum_{|u|=n-1} \lambda_{ux_0} Li_u + \sum_{|u|<n-1} \mu_u Li_u = 0, \\ 2i\pi \sum_{|u|=n-1} \lambda_{ux_1} Li_u + \sum_{|u|<n-1} \nu_u Li_u = 0. \end{cases}$$

By the induction hypothesis, we get the expected result.