Algebra and Algorithms for Differential Systems

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[summary by François Ollivier]

Abstract

This talk investigates algorithmic issues related to the formal resolution of algebraic differential systems, with a stress on the problem of testing components inclusion. Index reduction and applications to control theory are also considered.

News are also given of the `diffalg2` maple package which improves upon Boulier's work and will be part of a future Maple distribution.

1. Basic Algebraic Results

1.1. **Differential Algebras.** Details may be found in the classical book by Ritt [12], which remains an illuminating reference. The two first chapters provide a clear exposition of basic definitions and results. Some details on the low power theorem may be found in chapter 3. The book by Kolchin [9] is a reference book reserved to those having a good familiarity with the subject. Chapter 2 of Buium's book [3] is also a good introduction to differential algebra. The remaining chapters may be quite hard without a good previous knowledge of "modern" algebraic geometry, but contain many interesting new results. The paper [6], and thesis [8] contain details on the components problem. Details on Boulier's algorithm can also be found in [1, 2].

Differential algebra is a generalization of classical commutative algebra. We complete the ring structure with the datum of a set of mutually commuting derivations $\Delta = \{\delta_1, \ldots, \delta_n\}$. We may then define differential fields, modules and algebras in a straightforward way. A differential ideal of a differential ring A is an ideal I such that $\delta I \subset I$, for all $\delta \in \Delta$. Let A be a differential ring, and I be a differential ideal, then A/I has a natural structure of differential ring. The smallest differential ideal containing a set Σ is denoted by $[\Sigma]$.

We define differential polynomials in the following way: if A is a differential ring with derivation set Δ, Θ the free commutative monoid generated by Δ and X a set, the differential polynomial algebra $A\{X\}$ is the polynomial algebra $A[\Theta X]$ equipped with the only derivation set whose action restricted to A and ΘX is that of Δ.

Let A be a Ritt ring, i.e. a differential ring containing $\mathbb{Q}$. Then for every differential ideal $I \subset A$, the radical ideal $\sqrt{I}$ is differential. A differential ring A is radically Noetherian if for every set $\Sigma \subset A$ there exists a finite set B such that $\sqrt{[\Sigma]} = \sqrt{[B]}$. In the sequel, we will denote the perfect closure $\sqrt{[\Sigma]}$ by $\{\Sigma\}$.

Theorem 1 (Ritt-Raudenbush). *If A is radically Noetherian, then for all finite set X, $A\{X\}$ is radically Noetherian.*

[1] `http://daisy.uwaterloo.ca:80/~ehubert/Diffalg/`

Corollary 1. *Le I be a radical differential ideal, then I is a finite intersection of prime ideals $\cap_{i=1}^{r}\mathcal{P}_i$.*

1.2. **Differential Field Extensions.** Let $\mathcal{F}$ be a differential field, and $\mathcal{P}$ be a prime differential ideal of $\mathcal{F}\{X\}$, then the quotient ring $\mathcal{F}\{X\}/\mathcal{P}$ is a differential domain, and we can consider its fraction field K. So we can associate to any prime differential field $\mathcal{P}$ a differential field extension $K/\mathcal{F}$.

It is clear from the theorem above that a system of equations $\Sigma \subset \mathcal{F}\{X\}$ admits solutions in some field extension of $\mathcal{F}$ iff $\{\Sigma\} \neq 1$. So we need an algorithm to test if a system is consistent.

2. Algorithmic Tools

2.1. **Boulier's Algorithm.** Boulier's algorithm [2] is able to solve such problems as eliminating differential variables, and testing consistency of a differential system. It provides a description of the set of solutions as a finite union of algebraic quasivarieties, i.e. Zariski open subsets of differential algebraic varieties. Each of them is described by a characteristic set A (see [12] for a precise definition of this notion), according to a compatible ranking on the set of derivatives, and an inequation $h_A \neq 0$. Let u_P denote the greatest derivative of a polynomial P. The separant of P is $S_P := \partial P/\partial u_P$. As h_A is a multiple of the products of separants of polynomials in the characteristic set, the ideal $[A] : h_A^\infty$ is radical. Unlike Ritt's algorithms, Boulier's avoids factorizations for better efficiency. This is why it cannot return prime components.

Boulier's algorithm first proceeds by constructing an autocoherent set by repeated pseudo Euclidean reductions. An autocoherent set A being found, one need to test that it is the characteristic set of a radical differential ideal. According to *Rosenfeld's Lemma*, this may be reduced to an algebraic problem. We only have to test that A is a characteristic set of the algebraic ideal $(A) : h_A^\infty$. This may be done by computing a Gröbner basis of the ideal $(A, h_A w - 1)$, using an extra variable w and Rabinovich's trick.

2.2. **Singular Solutions and Inclusion of Components.** A difficult problem of differential algebra is to test whether two irreducible components defined by their characteristic sets are included one in the other. We are only able to test equality, and have necessary conditions, sufficient conditions, but no necessary and sufficient condition in the general case.

Consider a single polynomial equation: $P(t, y, \ldots, y^{(r)})$, where P is prime. The perfect ideal $\{P\}$ is a finite intersection of prime ideals, $\mathcal{P}_i$, associated to characteristic sets reduced to a single polynomial A_i. The general component A_1 is associated to P. The other correspond to essential prime components assuming that the $\mathcal{P}_i$ are not included one in the other.

Boulier's algorithm, like Ritt's algorithm, produces the characteristic sets A_i of singular components, but also characteristic sets B_j corresponding to the singular locus of the differential algebraic variety corresponding to the general solution. (Notice that, as we avoided factorizations, the A_i need not be prime and can represent more than one prime component.) The B_j and the A_i correspond to the solutions of the perfect ideal $\{P, S_P\}$. We have $\{P\} = \{P, S_P\} \cap \{P\} : S_P$. The solutions corresponding to non essential singular components are Zariski adherent to the regular place of the general component.

We may remark, that according to [11], determining the essential singular components is equivalent to finding a finite basis of $\{P\} : S_P$, i.e. to have an *effective* version of the Ritt-Raudenbush theorem.

2.3. **Some Effective Criteria of Inclusion.** For a differential equation of order 1, the singular solutions are envelopes of regular ones. E.g., for the equation $(y')^2 - 4y$, the solutions in the general component are parabolas $y(t) = (t+c)^2$, and the essential singular solution $y = 0$ is the envelope of these parabolas.

If we have a prime decomposition, we can obtain an algorithm for finding the minimal essential components of $\{P\}$ by using the low power theorem of Ritt.

Theorem 2. *The prime differential ideal $\{y\}$ is an essential component of $\{P\}$, iff the lower degree terms of P do not contain any strict derivative of y.*

From this, we deduce that $\{y\}$ is not an essential component of $y'^2 - 4y^3$. In such a case, the regular solutions are of the form $y(t) = 1/(t+c)^2$. When t goes to infinity, then y goes to 0. So the solution $y = 0$ is adherent to the the set of regular solutions. See [12, Chap. 6] for analytical versions of this adherence property.

The necessity proof relies of Levi's lemma which characterizes the monomials belonging to the differential ideal $[y^p]$ [12, Chap. 2], or on Kolchin's domination lemma. The sufficiency proof was obtained by Ritt, using a Puiseux series expansion.

In the case where we want to test the inclusion $\{P\} : S_P \subset \{Q\} : S_Q$, where $Q \neq y$, we need to find a *preparation polynomial*, i.e. a polynomial $M(z) = \sum_{\gamma=0}^{\ell} c_\gamma m_\gamma(z)$ such that c_γ does not belong to $\{Q\} : S_Q$, $CP = M(Q)$ and C is not a zero divisor modulo $\{Q\} : S_Q$. An algorithm is given to compute a preparation polynomial.

We also have a low power theorem for regular differential polynomials (see Hubert [7]). This theorem, together with Boulier's algorithm allows to find a minimal regular decomposition for $\{P\}$ without performing factorizations.

Theorem 3. *(Sufficiency) Let P be a non zero differential polynomial of $\mathcal{F}\{Y\}$, Q a square free polynomial. Assume that the preparation polynomial of P with respect to Q is $M = cz^p + R$, where $R \in [z]^{p+1}$, $p > 0$ and c is partially reduced with respect to Q. Then, $Q/\gcd(Q, c)$ is the characteristic set of an essential singular component of P.*

(Necessity) Under the same hypotheses, if the preparation polynomial is $M = c_0 z^p + \sum_{\gamma=1}^{\ell} c_\gamma m_\gamma + R$, where $R \in [z]^{p+1}$, the c_γ are partially reduced with respect to Q, then $Q/\gcd(Q, c_0, \ldots, c_\ell)$ is a characteristic set of a redundant component.

2.4. **Implementations.** The Rosenfeld-Gröbner algorithm of Boulier, implemented in the Maple package `diffalg`, has been improved with the new version `diffalg2`. Functions for computing preparation polynomials and finding initial components were added. It is available on the Web with a clear documentation, and an impressing set of examples.

3. Applications

3.1. **Control Theory.** Elimination in differential algebra allows to go from state-space to input-output representation by eliminating the state variables. It allows to test *observability* [4] and *identifiability* [5, 10].

Consider a system of the form $x_i' = P_i(x, u), y_j' = Q_j(x)$. To test observability, one has to compute a characteristic set for an ordering eliminating the variables x. The system is observable iff for each variable x_ℓ, the characteristic set contains a polynomial whose x_ℓ is the main derivative. Such a polynomial gives an implicit expression of x_ℓ as an algebraic function of the outputs y and the inputs or commands u *and their derivatives.* This makes such an expression of little applicability, due to the noise.

3.2. **Implicit Systems.** If we consider an implicit system $P_i(x', x) = 0$ where $\det(\partial P_i/\partial x_j')$, it is not possible to compute a power series or a numerical solution in a direct way. The system is not formally integrable. In fact, solutions, if any, do not exist for all initial conditions, and one may need first to determine the variety of compatible initial conditions. For this, one will need to differentiate the equation a number of time which is known as the *index* of the system. Computing characteristic sets, using the Rosenfeld Gröbner algorithm is a way of doing it.

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